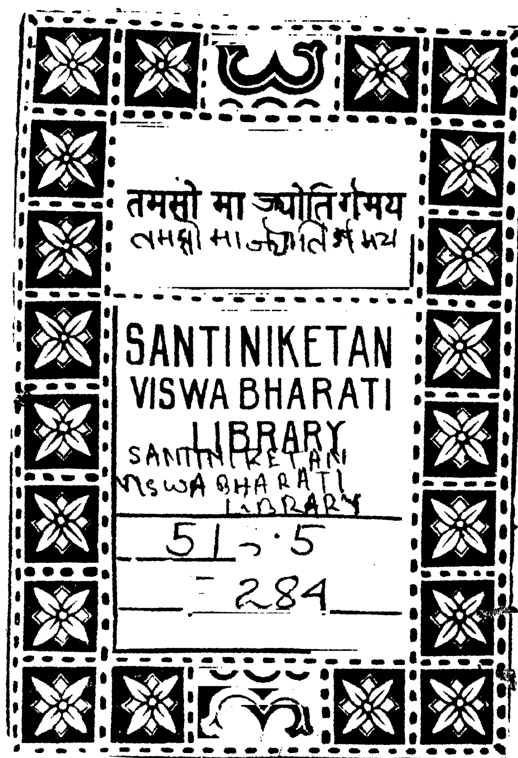


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CO-ORDINATE GEOMETRY

(PLANE AND SOLID)

FOR BEGINNERS

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CO-ORDINATE GEOMETRY

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FOR BEGINNERS

BY

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PREFACE

MOST of the existing text-books on Co-ordinate Geometry are written for the mathematical specialist, but there are many students, such as Engineers and Army candidates, who require only a working knowledge of the principles of elementary analysis and an acquaintance with the more important properties of the Conic Sections: this book is intended primarily to meet the needs of the latter class but it is hoped that it will prove suitable as a first course for those who are likely to study the subject more thoroughly afterwards.

In the early part of the book the use of formulæ is avoided, and methods are illustrated by numerical examples: at a later stage the proof of a formula is preceded by a numerical example in order to make clear the distinction between variables and constants, and to cultivate the power of solving problems even when the suitable formula is forgotten. Simple locus problems are frequently introduced.

Sufficient explanation of the methods and notation of the Calculus is given to enable the student to use it for finding gradients; he is thus enabled to deal at an early stage with curves which are not Conics.

Chapter X. contains, in addition to the usual equations for the plane and straight line in three dimensions, a short account of some of the more important curved surfaces.

The omission of such topics as Poles and Polars, Conjugate Points, Coaxal Circles, Harmonic Properties, has considerably narrowed the range of the examples, but every effort has been made to supply questions of an interesting type to illustrate the more straightforward applications of the subject.

CLIFTON,
December 1920.

CONTENTS

CO-ORDINATE GEOMETRY

CHAPTER I

Sign Convention.

The operations of addition and subtraction can be represented graphically in the following way :—

Suppose we require to show that $6+2=8$, we take any convenient length to represent one unit and mark off 6 of these lengths along a line starting from a point **O**, thus **OA** represents 6 units. Proceeding in the same direction from



FIG. 1

A we mark off two more units, **AB**, and obtain **OB**$=8$ units.

If, however, we wish to show that $6-2=4$, after marking off 6 units to the right from **O** we must now reverse the

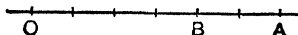


FIG. 2

direction and mark off 2 units to the left from **A**, thus reaching **B**, where **OB**$=4$. If, after marking **OA**$=6$ we mark off 8 units to the left, we arrive at a point **B′**, 2 units

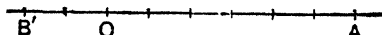


FIG. 3

to the *left* of **O**, and since $6-8=-2$, it follows that **OB′** represents -2.

Similarly, if we mark off positive units in the direction **OD**, we must represent negative numbers by moving in the opposite direction **OE**, so that if **OD** represents +5, then **OE** represents −3.

If we take two axes, **OX** and **OY**, at right angles to one another it has been agreed that the directions **OX** and **OY** shall be considered the positive directions. This agreement is called the sign convention.

D
O
E
Fig. 4

It has been shown that **OX′** and **OY′** will necessarily be the directions for representing negative numbers.

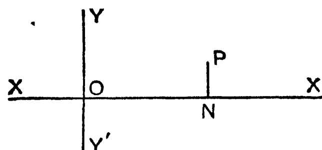


Fig. 5

Co-ordinates.

If, through any point **P** on the paper, we draw a line parallel to the axis **OY**, cutting **OX** at **N**, we know the position of **P** when we know **ON** and **NP**. These lengths are called the co-ordinates of the point, **ON** is called the abscissa and **NP** the ordinate.

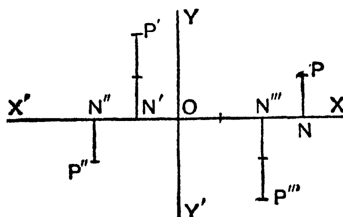


Fig. 6

The **X** co-ordinate (**ON**) or abscissa is always stated first, thus the co-ordinates of **P** are (3, 1), and of **P′** (−1, 2), of **P″** (−2, −1), of **P‴** (2, −2).

The co-ordinates of **O**, called the origin, are (0, 0).

Length of a Line joining Two Points.

Let the co-ordinates of P be (2, 2) and of Q (5, 3).

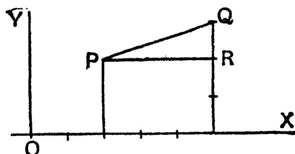


FIG. 7

If through P we draw PR parallel to OX we see from the figure that PR=3 and RQ=1.

$$\therefore \ PQ=\sqrt{3^2+1^2}=\sqrt{10}$$

If P is (−3, 2) and Q is (4, −3), we see from a figure if QR

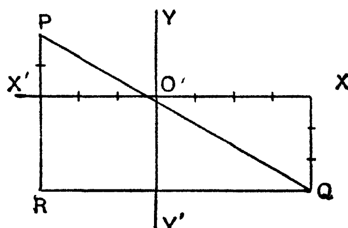


FIG. 8

is drawn parallel to OX that QR=7 and PR=5.

$$\text{Hence } PQ=\sqrt{49+25}=\sqrt{74}$$

Co-ordinates of a Point dividing a Line in a Given Ratio.

Let P (1, 2,) Q (4, 3) be two points. It is required to

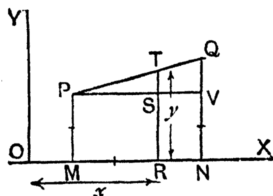


FIG. 9

find the co-ordinates of T which divides PQ in the ratio

$$\frac{PT}{TQ}=\frac{2}{1}$$

Let $(x\,y)$ be the co-ordinates of T. From the figure we have the similar triangles TSP, QVP, where $TS=y-2$, $PS=x-1$, $QV=1$, $VP=3$.

Now
$$\frac{TS}{QV}=\frac{TP}{QP}=\frac{2}{3}$$

i.e.
$$\frac{y-2}{1}=\frac{2}{3} \qquad \therefore\ y=2\tfrac{2}{3}$$

Also
$$\frac{SP}{VP}=\frac{TP}{QP}=\frac{2}{3} \qquad i.e.\ \frac{x-1}{3}=\frac{2}{3} \qquad \therefore\ x=3$$

$\therefore$ co-ordinates of T are 3 and $2\frac{2}{3}$

EXAMPLES I

1. State the distance between the following pairs of points on the axes :—

$x_1=3,\ x_2=-2$; $x_1=-5,\ x_2=-2$; $y_1=5,\ y_2=-3$

2. Find from a figure the distance between P and Q in the following cases :—P (3, 1), Q (1, 2) ; P (−3, 1), Q (1, −2) ; P (−3, −1), Q (−1, 2).

3. From a figure find the co-ordinates of the mid point of the line joining PQ in the following cases :—P (1, 2), Q (3, 6) ; P (−1, 2), Q (3, 6).

4. Plot the graphs of the lines (i.) $y=3x$; (ii.) $y=3x+2$; (iii.) $y=3x-2$. What connection is there between these three lines ?

5. Prove that the triangle ABC is isosceles given A (1, 4), B (4, 1), C (8, 8).

6. Prove that the points (−4, −2) (2, 0) (8, 6) (2, 4) are the angular points of a parallelogram.

7. Find the distance of the point P whose co-ordinates are (x, y) from the point (0, 1). If this distance is equal to 4, find the relation between x and y.

8. Write down the distance of the point P (x, y) from A (−2, 0) and from B (2, 0). If $PA=PB$, find the relation between x and y. Explain your result.

9. The angular points of a triangle are **A** (—2, 0), **B** (0, 4), **C** (2, 0). If **D** is the mid point of **BC**, find the length of **AD**.

10. Find the lengths of the medians of the triangle **A** (2, 0), **B** (4, 4), **C** (6, 2).

11. Find the co-ordinates of the centroid of the triangle **A** (0, 2), **B** (3, 5), **C** (5, 1). (The centroid **G** divides the medians in the ratio 2 : 1 measured from the vertices.)

12. Write down the distance of **P** (x, y) from the point (2, 0). If this distance equals the distance of **P** from the **Y** axis, prove that $y^2=4(x-1)$.

Plotting Graphs.

The student is probably familiar with the process of plotting the graph of an equation such as $y=x-2$, but as it is important to have a clear idea of what is being done we will consider the process in detail.

Taking various values of x, say from -2 to $+3$, we calculate the corresponding values for y and obtain the following table :—

x	-2	-1	0	1	2	3
y	-4	-3	-2	-1	0	1

We now take each pair of values as the co-ordinates of a point—*e.g.* P_1 (—2 —4), P_2 (—1, —3), etc., and plot these points on squared paper.

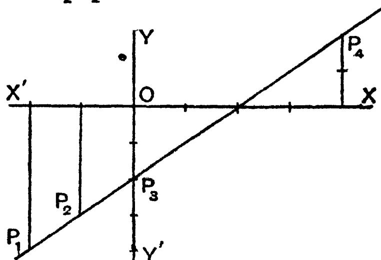


FIG. 10

Joining up the points we get a straight line called the graph of the equation.

NOTE.—(i.) The corresponding values for x and y in the above table satisfy the given equation.

(ii.) The co-ordinates of ANY point on the graph will satisfy the equation—*e.g.* if we take the point P_4 on the line we see from the graph that its co-ordinates are (4, 2), and $x=4$, $y=2$ satisfy the equation $y=x-2$.

(iii.) In the equation $y=x-2$, x and y represent the co-ordinates of any point on the graph, and the values of these co-ordinates satisfy the equation.

Graph of $y=x^2$.

Taking values of x from -3 to $+3$ we obtain the following table:—

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

We now plot the points whose co-ordinates are $(-3, 9)$ $(-2, 4)$, etc., and obtain the curve in Fig. 11.

Notice again that the co-ordinates of *any* point on this graph satisfy the relation $y=x^2$, *e.g.* the co-ordinates of P are $ON=1{\cdot}5$ and $PN=2{\cdot}25$; also $PN=ON^2$ for $2{\cdot}25=(1{\cdot}5)^2$.

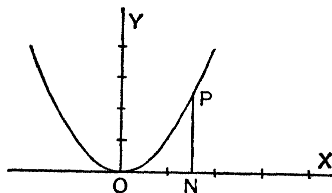


FIG. 11

We therefore arrive at the definition of the *Equation of a Curve* as the relation between the co-ordinates x and y of *any point* on the curve.

This definition should be carefully remembered.

In co-ordinate geometry we are given the law to be obeyed by a moving point and we are required to find the equation of the curve described by the point which moves.

according to this law. The path of the point is known as its *Locus.*

Gradient of a Straight Line.

P Q R are three points on a straight line whose ordinates are **PN**, **QM**, **RS** respectively. The gradient of the line means the increase in the ordinate for unit increase in the abscissa.

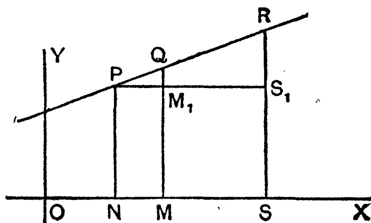


FIG. 12

In passing from **P** to **Q** the ordinate increases by QM_1 while the abscissa increases by $NM=PM_1$.

∴ increase of ordinate for unit increase in abscissa

$$=\frac{QM_1}{PM_1}=\text{gradient of the line}$$

If we travel from **P** to **R** the gradient is $\frac{RS_1}{PS_1}$, which by similar triangles equals $\frac{QM_1}{PM_1}$, and it is clear that the gradient is constant for all points on the line.

If the units on both axes are of the same length this ratio represents the trigonometrical tangent of the angle between the line and the positive direction of the **X** axis.

When plotting values showing the relation between physical quantities it is often convenient to take different lengths for units on the two axes—*e.g.* when plotting values satisfying $v=32t$, where v is the velocity in ft.-sec. and t the time in secs. we might take 1 division along **OX** to represent 1 sec. and 1 division along **OY** to represent 10 ft.-sec. In co-ordinate geometry it is necessary to have the same units on both axes if we are to draw the correct shape of a curve.

Equation of a Straight Line.

(1) *Passing through the Origin.*—Let the gradient of the

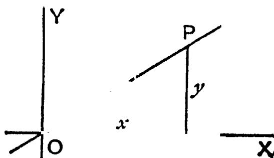


FIG. 13

required line be $\frac{2}{3}$, we have to find the relation between x and y, the co-ordinates of any point P on the line.

Take any point P (x, y) on the line. The gradient is $\frac{y}{x}$ and this has to equal $\frac{2}{3}$

$$\therefore \frac{y}{x}=\frac{2}{3} \text{ or } y=\frac{2x}{3} \text{ is the required equation}$$

Similarly if the gradient is to be m we have for the equation $y=mx$.

N.B—If we plot out all the points for which $x=o$ we get the line OY and if we plot all the points for which $y=o$ we get the line OX $\therefore$ the equation of OX is $y=o$ and of OY it is $x=o$

(2) *Not passing through the Origin.*—Let the gradient of the straight line be $\frac{2}{3}$ and the intercept it makes on OY be 1.

Let P (x, y) be any point on the line. Through R draw

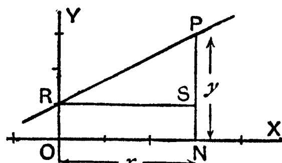


FIG. 14

RS parallel to OX. The gradient of the line is $\frac{PS}{RS}=\frac{y-1}{x}$.

This has to equal $\frac{2}{3}$.

$\therefore \frac{y-1}{x}=\frac{2}{3}$ or $y=\frac{2x}{3}+1$ is the required equation

If we plot the position of all points for which $x=2$ we get a straight line parallel to OY. Hence the equation $x=a$ when a is a constant, represents a straight line parallel to OY. Similarly $y=b$ will represent a straight line parallel to OX.

Equation of a Circle.

(1) *Centre at the Origin.*—Let 2 be the radius of the circle. We have to find the relation between x and y, the co-ordinates of any point P on the circle. The condition to be

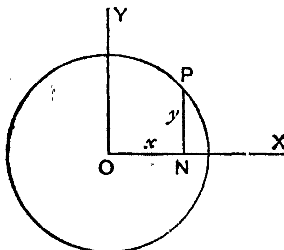


FIG. 15

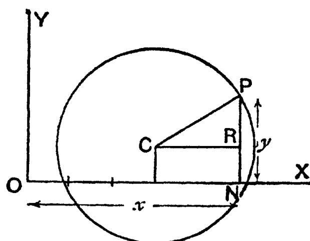


FIG. 16

satisfied is that P must always be at a distance 2 from the centre O.

Now $OP^2=x^2+y^2$ and $OP^2=4$

$$\therefore x^2+y^2=4 \text{ is the required equation}$$

N.B.—It is convenient to take P so that x and y are both positive.

(2) *Centre at the Point* (3, 1).—Let P (x, y) be any point

on the curve, then from the figure PR$=y-1$, and CR$=x-3$. Also CP$=2$.

$$\therefore \text{CP}^2=(x-3)^2+(y-1)^2=4$$

$$\therefore (x-3)^2+(y-1)^2=4 \text{ is the required equation}$$

Equation of a Parabola.

A parabola is a curve described by a point which moves so that its distance from a fixed point equals its distance from a fixed line.

The equation will depend on the position of the fixed point and line with reference to the axes OX and OY.

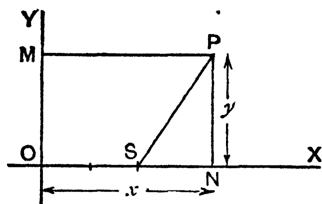


Fig. 17

Suppose we take OY to be the fixed line and take S the fixed point on OX at a distance 2 from O.

Let P (x, y) be any point on the curve, then if PM is the perpendicular on OY we are given that SP$=$PM.

Now SN$=x-2$ and PN$=y$.

$$\therefore \text{SP}^2=(x-2)^2+y^2. \quad \text{Also PM}=x$$

$$\therefore (x-2)^2+y^2=x^2$$

$$\therefore y^2=4x-4=4(x-1) \text{ is the required equation}$$

EXAMPLES II

Find the equations of the locus of P (x, y) in the following cases :—

1. Distance of P from the Y axis$=2$.
2. Distance of P from the origin $=4$.
3. Distance from X axis is twice its distance from the Y axis.
4. Distance from the origin equals twice its distance from the Y axis.

5. Distance from (1, 2) equals twice its distance from the X axis.

6. Distance from (−1, 2) equals its distance from (4, 4).

7. Distance from origin equals twice its distance from (2, 1).

8. Distance from the point (1, 2) equals 3.

9. Distance from (1, 2) equals its distance from the line $x=-2$.

10. Distance from (2, 2) equals its distance from the line $y=4$.

11. The sum of its distances from OX and OY=2.

12. Distance from (−1, 0) is to its distance from (+1, 0) in the ratio 2 : 1.

13. The sum of the squares of its distances from (3, 0) and (−3, 0) equals 68.

14. Distance of P from (8, 0) : distance of P from (2, 0) in the ratio 2 : 1.

15. Find the equation of a line making an intercept of −2 on OX and +3 on OY.

16. Find the equation of the line through the origin with gradient $\frac{3}{1}$.

17. Find the equation of the line having a gradient $\frac{2}{1}$ which makes an intercept of 1 on the Y axis.

18. Find the equation of a line having a gradient $\frac{4}{1}$ which makes an intercept of 2 on the X axis.

Equation of a Line.

(1) *Having a Given Gradient and passing through a Given Point.*—Let the gradient be $\frac{1}{2}$ and the given point be A (1, 3).

3. Find the points of intersection of the following:—(i.) $2x+y-4=0$, $x-y+2=0$; (ii.) $x+y=1$, $x^2+y^2=4$; (iii.) $y=2x$, $y^2=4x$.

4. Find where the line through (2, 4) whose gradient is 2 cuts the line $3x+y+1=0$.

5. Find the equation of a straight line through (2, 1) parallel to $y=3x$.

6. Find the equations of the lines joining the origin to the points where the curves $y^2=\frac{9x}{4}$ and $x^2+y^2=25$ intersect.

7. Without drawing a figure, determine which of the following points are on the line $2x+y=4$:—(1, 0), (1, 3), (2, 0), (3, −2), (0, 3).

8. Without drawing a figure, determine which of the following points are on the curve $y^2=2(x-2)$:—(0, 2), (−1, 0), (4, 2), (11, 3).

9. Find whether the point of intersection of $x-2y+4=0$ and $2x+y+6=0$ lies on the line $x+4y+1=0$.

10. Find the equation of the line joining the points of intersection of $2x+y=4$ with $x-y+1=0$ and $2x-y-1=0$ with $x+y-8=0$.

11. The line $2x+y+4=0$ cuts the X axis at A. What is the ordinate of A? Find its abscissa. Find the co-ordinates of B where this line cuts OY.

12. Find the co-ordinates of the points where the line joining P (2, 1) Q (4, 3) cuts the two axes.

13. The line AB whose equation is $3x+y=6$ cuts the axes at A and B. Find the distance AB.

14. From a point P (x, y) perpendiculars PM, PN are drawn to the axes. If P moves so that the area ONPM is always 10, find the equation of the locus of P and sketch its graph.

15. If the curve $\frac{x^2}{4}+\frac{y^2}{9}=1$ cuts OX at A and OY at B, find the distance AB.

CHAPTER II

USE OF FORMULÆ

HITHERTO we have investigated only the equations of lines with positive gradients.

Considerable confusion may arise with regard to signs when an equation involving negative quantities is to be deduced by first principles from a figure.

Such difficulties are avoided in co-ordinate geometry by obtaining a formula for the case in which all quantities are positive. By adopting the sign convention we are able to use these formulæ for all cases, as will appear in the course of this chapter.

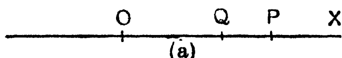


FIG. 21 (b)

If two points, P and Q, are on the X axis, and $OP=x_1$, $OQ=x_2$, the length QP will be x_1-x_2. (Fig. 21 (a).)

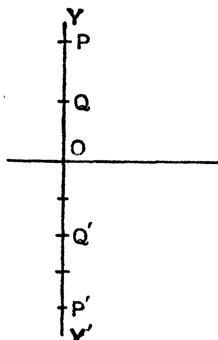


FIG. 22

If P and Q are on opposite sides of the origin the length QP is still x_1-x_2, provided we give x_2 its proper sign when we substitute its numerical value. If, for instance, $OP=x_1=3$ and $OQ=x_2=-2$ then $x_1-x_2=3-(-2)=5=QP$. (Fig. 21 (b).)

Similarly if OP is y_1 and OQ is y_2, then $PQ=y_1-y_2$. (Fig. 22.)

If $y_1=-4$ and $y_2=-2$, then $y_1-y_2=-4-(-2)=-2=P^1Q^1$.

Length of Line joining Two Points.

If the two points are **P** $(x_1\ y_1)$ and **Q** $(x_2\ y_2)$ it is clear from the figure that $PQ^2=(x_2-x_1)^2+(y_2-y_1)^2$

$$\text{or } PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

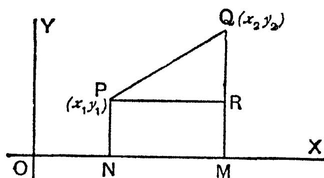


FIG. 23

Suppose $x_1=-3$, $y_1=2$ and $x_2=4$, $y_2=-3$, we get from the formula

$$\begin{aligned} PQ^2&=\{4-(-3)\}^2+\{-3-2\}^2\\ &=(4+3)^2+(-5)^2\\ &=49+25=74 \end{aligned}$$

$$\therefore\ PQ=\sqrt{74}$$

This is the same result as we obtained from a figure on p. 3.

Co-ordinates of a Point dividing a Line in a Given Ratio.

To find the co-ordinates **X**, **Y** of the point **R** on the line joining **P** $(x_1\ y_1)$ and **Q** $(x_2\ y_2)$ when $\dfrac{PR}{RQ}=\dfrac{l}{m}$

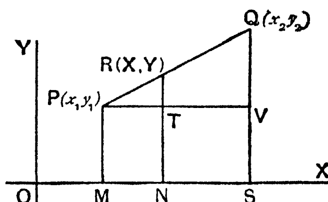


FIG. 24

draw **PTV** parallel to **OX**. Then by similar triangles **RTP**, **QVP** we have $\frac{TP}{VP}=\frac{RP}{QP}=\frac{l}{l+m}$

i.e.
$$\frac{X-x_1}{x_2-x_1}=\frac{l}{l+m}$$
$$\therefore\ X=\frac{lx_2+mx_1}{l+m}$$

Similarly since $\frac{RT}{QV}=\frac{RP}{QP}$ we find $Y=\frac{ly_2+my_1}{l+m}$.

If $l=m$, **R** is the mid-point of **PQ** and $X=\frac{x_1+x_2}{2}$, $Y=\frac{y_1+y_2}{2}$.

In this case it is clear that **ON** is the Arithmetic Mean between **OM** and **OS** and that **RN** is the Arithmetic Mean between **PM** and **QS**.

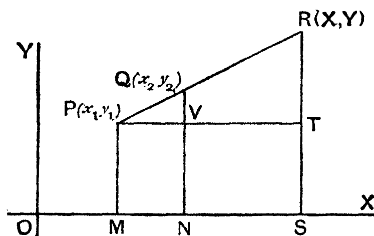


FIG. 25

If **R** divides **PQ** externally in the ratio $l:m$—*i.e.* **PR** : **RQ** $=l:m$ (Fig. 25), show from the relation $\frac{TP}{VP}=\frac{RP}{QP}=\frac{l}{l-m}$ that
$$X=\frac{lx_2-mx_1}{l-m}$$

Similarly $Y=\frac{ly_2-my_1}{l-m}$.

Equation of a Straight Line.

(1) *Through the Origin.*—If we plot the graph of $y=-2x$ we have the line **AOB**. (Fig. 26.)

Here the angle **AOX** (θ) is obtuse and $\tan\theta = -2$.

The equation of the line is still of the form $y=mx$ where m represents the tangent of the angle between the line and the *positive direction* of the **X** axis.

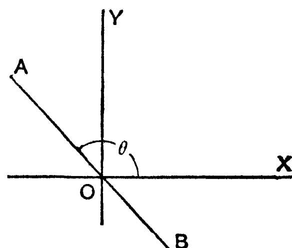


FIG. 26

(2) *Gradient m. Intercept on* **OY**$=b$.—We draw the figure for m and b both positive.

If **P** (x, y) is any point on the line **RP**, and **OQ** is drawn

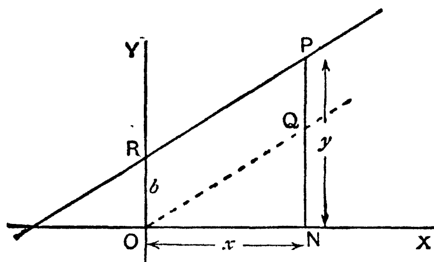


FIG. 27

parallel to **RP** through the origin, we have, since m is the gradient of both **RP** and **OQ**, **QN**$=mx$.

But $y=$ **QN**$+$**PQ**$=$**QN**$+$**RO**$=mx+b$.

$\therefore$ $y=mx+b$ is the required equation

When two quantities, x and y, vary so that the ratio between them is constant we have $\frac{y}{x}=k$, *i.e.* $y=kx$. Here y is said to vary directly as x. If x is doubled, y is doubled and so on. The graph showing the relation between x and y is a straight line through the origin.

Suppose a quantity y is made up of two parts, y_1 and y_2, one of which (y_1) is constant and the other (y_2) varies as x.

Let y_1 be represented by the constant b, then y_2 will be such that $\frac{y_2}{x}$ is a constant (say m)—*i.e.* $y_2=mx$.

$$\therefore\ y=y_1+y_2=b+mx$$

The graph will be a straight line whose intercept on the Y axis is b.

EXAMPLE. Write down the equation of a line whose gradient is $-\frac{1}{2}$ which makes an intercept -2 on OY.

The equation is $y=mx+b$ where $m=-\frac{1}{2}$; $b=-2$.

Hence $y=-\frac{1}{2}x-2$ or $x+2y+4=0$.

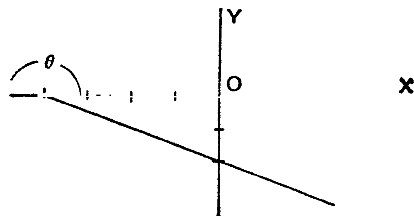


FIG. 28

Plotting this on squared paper we see that $\tan\theta=-\frac{1}{2}$ and $b=-2$.

If we plot out the graph of the line $2x+y-4=0$, we do so most readily by putting $x=0$ then $y=4$, and $y=0$ then $x=2$. Hence the line joining (0, 4) and (2,0) is the required graph. $m=\tan\theta$ in this case is -2, and the intercept on the Y axis is 4.

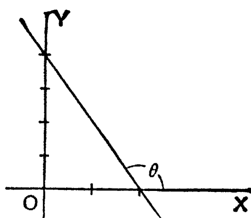


FIG. 29

This may be seen at once by writing $2x+y-4=0$ in the form $y=mx+b$, *i.e.* $y=-2x+4$.

$$\therefore\ m=-2,\ b=4$$

Similarly we see for $3x-2y+6=0$ that $m=\frac{3}{2}$ and $b=3$, for it may be written $y=\frac{3x}{2}+\frac{6}{2}$.

General Equation of a Straight Line.

When m or b are fractions as in the equation $y=\frac{3x}{2}+3$, on multiplying throughout by 2 and bringing all terms to the left-hand side we shall obtain the equation $3x-2y+6=0$. Writing a for the coefficient of x, b for the coefficient of y, and c for the numerical term, we have $ax+by+c=0$, which is called the general equation of the straight line.

Equation of the Line through a fixed point A (x_1y_1) whose Gradient is m.

If the co-ordinates of the given point A are $x_1\ y_1$, and the gradient is m, then $\frac{y-y_1}{x-x_1}=m$.

$\therefore$ **$y-y_1 \quad m\,(x-x_1)$ is the required equation of the line.**

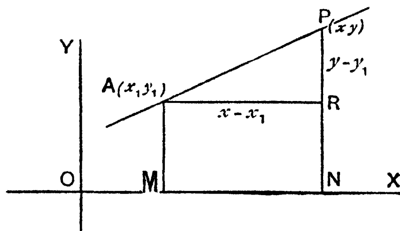


FIG. 30

Note that $x\ y$ are used for the co-ordinates of *any* point on a curve ; x_1y_1, x_2y_2 for fixed points.

Much difficulty is avoided by using the general form

of the equation in all cases, although the figure above is drawn for the case in which x_1y_1 and m are positive.

For instance, to find the equation of the line through $(-2, -3)$ whose gradient is -2 we have $y-(-3)=-2\{x-(-2)\}$ or $y+3 \quad -2(x+2)$.

It is at once clear that this equation is satisfied by the point $(-2, -3)$, and that its gradient is -2, but considerable confusion would arise in determining the equation from a figure.

This equation is of great importance when finding the equations of tangents and normals to curves.

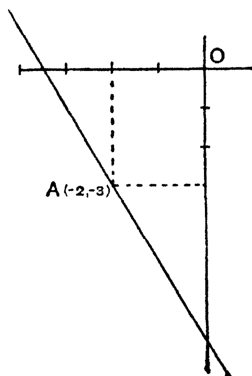


FIG. 31

EXAMPLE. Find the equation of the line through (4, 3) parallel to $2x+3y-3=0$.

The gradient of the given line is $-\frac{2}{3}$, as is seen by writing it in the form $y=-\frac{2}{3}x+1$.

. Since the required line is to go through (4, 3) and have a gradient $-\frac{2}{3}$ its equation will be $y-3=-\frac{2}{3}(x-4)$ or $2x+3y-17=0$.

Alternative Form.

If the slope of the line is θ the equation of the line through A (x_1y_1)—viz. $y-y_1=m(x-x_1)$ may be written

$$y-y_1=\tan\theta\,(x-x_1) \quad \text{or} \quad \frac{y-y_1}{\sin\theta}=\frac{x-x_1}{\cos\theta}$$

If r is the distance from A (x_1y_1) to any point P along the line

$$x-x_1=r\cos\theta \text{ and } y-y_1=r\sin\theta$$

$$\therefore \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

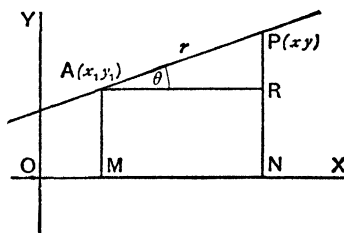


Fig. 32

Example. Find the distance from the point (5, 1) on the line $y-x+4=0$ to the point of intersection of this line and the line $2x+y=10$.

If θ is the slope of the line $y-x+4=0$ which may be written $y=x-4$, we see that $m=\tan\theta=1$.

$$\therefore \sin\theta=\frac{1}{\sqrt{2}} \qquad \cos\theta=\frac{1}{\sqrt{2}}$$

If r is the distance of any point P (x, y) from (5, 1) along this line the co-ordinates of P may be written

$$x=5+r\cos\theta \qquad y=1+r\sin\theta$$

If this point is to lie on $2x+y=10$ we have

$$2\left(5+r\frac{1}{\sqrt{2}}\right)+\left(1+r\frac{1}{\sqrt{2}}\right)=10$$

$$\therefore r\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)=10-1-10$$

$$\therefore r=-\frac{\sqrt{2}}{3}=-{\cdot}47$$

A figure will show the significance of the negative sign.

Equation of a Line joining Two Fixed Points.

Let A (x_1y_1), B (x_2y_2) be the two fixed points, and let P (x, y) be any point on the line.

Since A and B are fixed, the gradient of the line is known, and it is $\frac{y_2-y_1}{x_2-x_1}$.

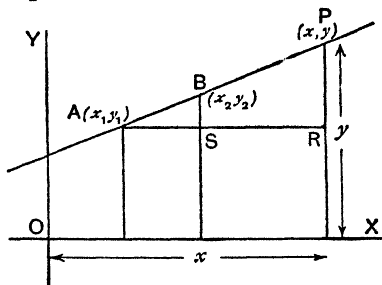


FIG. 33

The equation of any line through (x_1y_1) having a gradient m is $y-y_1=m(x-x_1)$. Hence for the required line the equation is $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$.

This result is more easily remembered in the form

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

Note that y_1 comes below y_1 and x_1 below x_1.

EXAMPLES IV

1. State the gradient and the intercept on the Y axis of the following lines :—(i.) $y=-2x+3$; (ii.) $2y=x+6$; (iii.) $x+2y-4=0$; (iv.) $3x-2y-6=0$.

2. Write down the equations of the lines whose gradient and intercept on the Y axis are : (i.) $m=2$, $b=1$; (ii.) $m=-2$, $b=1$; (iii.) $m=-2$, $b=-1$; (iv.) $m=\frac{1}{2}$, $b=\frac{3}{2}$.

3. Find the equations of the lines joining the following pairs of points, and state the gradient of each :—(i.) (2, 0) (0, 3) ; (ii.) (—2, 0) (0, 3) ; (iii.) (—2, 0) (0, —3).

4. Write down the equations of the following lines given the angle θ they make with the X axis and the intercept b on

the **Y** axis:—(i.) $\theta=30°$, $b=2$; (ii.) $\theta=60°$, $b=1$; (iii.) $\theta=120°$, $b=1$; (iv.) $\theta=120°$, $b=-1$.

5. Write down the equations of the lines through the point **P** which have a gradient m when: (i.) **P** (2, 3) $m=2$; (ii.) **P** (−2, 3) $m=2$; (iii.) **P** (−2, −3) $m=2$; (iv.) **P** (−2, −3) $m=-2$; (v.) **P** (2, −3) $m=-2$; (vi.) **P** (4, 2) $m=\frac{1}{2}$; (vii.) **P** (4, −2) $m=-\frac{1}{2}$.

6. Write down the equations of the lines through the points **P** and **Q** when (i.) **P** (0, 0) **Q** (2, 3); (ii.) **P** (−1, 2) **Q** (3, 4); (iii.) **P** (2, −4) **Q** (5, −6); (iv.) **P** (2, 0) **Q** (4, −2); (v.) **P** $\left(1, \frac{1}{2}\right)$ **Q** $\left(-2, -\frac{1}{2}\right)$. Plot these lines and find their gradients.

7. Show that $\frac{x}{2}+\frac{y}{3}=1$ represents a line whose intercepts on the axes are 2 and 3. What is the corresponding equation if the intercepts are a and b?

8. Find the intercepts on the axes made by the line $3x+4y=12$. Draw a perpendicular **ON** to this line from the origin and calculate its length.

9. Find the gradients of the lines $y=2x+3$ and $y=x+4$. If α is the angle between the lines, calculate $\tan\alpha$ from your results and find α in degrees.

10. Find the co-ordinates of the point **M** on the line joining **P** (2, 3) **Q** (4, 7) when (i.) **PM**=**MQ**; (ii.) **PM**=2**MQ**; (iii.) 2**PM**=**MQ**. Also when **P** (−2, 3) **Q** (4, −7) given (i.) **PM**=**MQ**; (ii.) **PM**=2**MQ**; (iii.) **PM**=3**MQ**.

11. Find the co-ordinates of **M** on **PQ** produced given **P** (2, 3) **Q** (4, 7) when (i.) **PM**=2**QM**; (ii.) **PM**=3**QM**.

12. Write down the equations of lines whose gradient is m and intercept on the **X** axis is a where (i.) $m=1$, $a=2$; (ii.) $m=\frac{1}{2}$, $a=1$; (iii.) $m=-2$, $a=-2$.

13. Find the equations of the sides of the triangle **ABC** given **A** (2, 1), **B** (−3, −2), **C** (3, −2).

14. Find the equations of the lines joining the vertices to the mid-points of the opposite sides in the triangle **ABC** (qn. 13) and find the co-ordinates of **G**, their point of intersection. Verify that **G** divides the medians in the ratio 1 : 2.

15. Find the equation of a line through (2, 1) parallel to $2x+y+4=0$.

16. Find the equations of a line making equal positive intercepts on the axes and passing through the point (1, 3).

17. Find the equations of the lines joining the origin to the two points of intersection of $y=3x+2$ and the circle $x^2+y^2=4$.

18. The expenses of a holiday are partly constant (*e.g.* the railway fare) and partly vary as the number of days. If the fare is £5 and the cost per day is £1, 10s, write down the cost (c) of a holiday of n days. What kind of graph represents the relation between c and n? What is the gradient of the graph if n is measured along **OX**.

19. The law of a machine is $P=aW+b$ where **P** is the effort and **W** the load in lb. Sketch the graph showing the relation between **P** and **W**.

Given

P	60	75	100	125	145
W	230	300	430	560	660

calculate a and b and find **P** when $W=500$.

20. A man walks s miles at v miles per hour. Write down the relation between s, v and t where t is the time taken in hours. What kind of graph shows the relation between s and t? What is the gradient of the graph (t measured along **OX**)? If s is measured from a point **A** and t is measured from the moment the man passed **B**, which is b miles beyond **A**, write down the relation between s and t. What kind of graph will this give? What is its gradient and what is the intercept on the **Y** axis if t is taken along **OX**?

21. **A** and **B** are the tops of two poles, 5 feet apart, whose heights are 3·5 feet and 7·8 feet respectively. Find by writing down the equation of the line **AB** how far from the foot of **A** it would, if produced, meet the ground. Also find the height of a point on this line whose horizontal distance from **A** in the direction **AB** is 12 feet.

22. A horizontal straight pipe is laid beneath the floor of a room **OPQR** and passes through the wall **OP** at **A** and **OR** at **B**, 3 feet and 5 feet respectively from the corner **O**. Find by analysis where it will pass through the walls **QP** and **QP** if **OP**=15 feet and **OR**=12 feet.

23. A railway passes through a station 5 miles N. of a place **O** and runs NE. Another line passing through a place 10 miles E. of **O** runs NW. Find by analysis how far north and how far east of **O** the two lines cross.

24. The two perpendicular edges of a set square are **OA**= 6 inches and **OB**=3 inches. A line parallel to **OB** and $2\frac{1}{2}$ inches from it is drawn on the set square. Find its length by using the equation of the line **AB**.

25. Find the distance from (1, 2) measured along the line $3x-4y+5=0$ to the point where this line cuts $2x+y=6$.

26. Find the distances from $(-1, 1)$ measured along the line $4x+3y+1=0$ to the points where this line cuts the circle $x^2+y^2=4$.

27. If the curves $x^2+y^2=100$ and $2y^2=9x$ intersect at **P** and **Q**, find the co-ordinates of the mid-point of the line **PQ**.

28. Find the length of the common chord of $x^2+y^2=13$ and $y^2=3x+3$.

29. Find the length of the line $x-7y+25=0$ intercepted by $x^2+y^2=25$.

30. **ABCD** is a rectangular field **AB**=150 yards, **BC**=100 yards. A cord is pegged from a point **P** in **AB**, 10 yards from **A** to a point **Q**, 20 yards from **AD** and 15 yards from **AB**. Find **DR** if this line when produced cuts the other side of the field at **R**. Where would a line through **A** to the mid-point of **BC** cut **DC** ?

31. Two poles **AB**, **CD** are 20 feet apart. From **A**, 5 feet from the ground a cord runs to **C**, 20 feet from the ground. Find by analysis how far from **B** this cord, if produced, would meet the ground. If another cord is run from **B** to a point 10 feet above **C**, at what height from the ground would the two cords meet ?

32. A straight railway runs from **A**, 10 miles N. of **B**, to **C**, 20 miles E. of **B**. It is crossed at **D** by another line running

from a place 5 miles S. of **B** to a place 10 miles N. of **C**. How far from **B** is the junction?

33. A diagonal **AC** is drawn on a rectangle **ABCD**. **AB**= 10 inches, **BC**=8 inches. Another line is drawn from the mid-point of **AB** to **D**. How far from **AB** and from **AD** is the point of intersection of the lines?

Plotting of Curves.

A general idea of the shape and position of a curve may often be obtained from a rough graph, and in such cases the points illustrated in the following examples should be carefully noted :—

(1) *Plot the curve* $y=x^2$.

Note.—1. Since x^2 is positive no negative values of y are possible.

2. For any value of y there will be two equal values of x of opposite sign. The curve is therefore symmetrical about the **Y** axis.

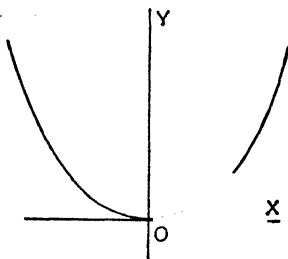


Fig. 34

3. The equation is satisfied by (0, 0). The curve therefore goes through the origin.

(2) *Plot the curve* $xy-2y-4=0$.

Note.—1. Solving for y we have $y=\dfrac{4}{x-2}$.

When x is large $+^{ve}$, y is small $+$.
When x is large $-^{ve}$, y is small $-^{ve}$.
When x approaches 2, y becomes very large.
When x is slightly greater than 2, y is $+$.
When x is slightly less than 2, y is $-^{ve}$.

2. Solving for x we have $x=2+\dfrac{4}{y}$.

When y is large $+^{ve}$, x is slightly greater than 2.

When y is large $-^{ve}$, x is slightly less than 2.

If the line $x=2$ is drawn we see that as y increases, the

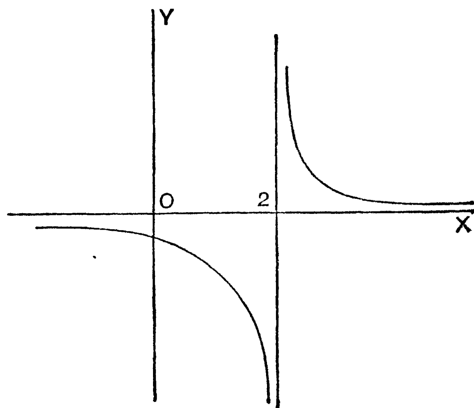


Fig. 35

curve continually approaches the position of the line. Such a line is called an asymptote of the curve.

3. When $x=o$, $y=-2$.

When y approaches o, x becomes very large. $y=o$ is an asymptote.

EXAMPLES V

1. State between what values of x the following expressions are (1) positive, (2) negative:—(i.) $(x-1)(3+x)$; (ii.) $(x-1)^2(x+3)$; (iii.) $(x-2)^2(x^2-9)$. Draw rough graphs.

2. Between what values of x is y positive, given

$$y=(x+2)\left(\frac{3}{2}-x\right)$$

3. Draw rough graphs of the following curves:—(i.) $xy=4$ (a rectangular hyperbola); (ii.) $y=x^3$ (a cubical parabola); (iii.) $x^2+4y^2=16$ (an ellipse); (iv.) $y^2-4x+16=0$ (a parabola); (v.) $y=(x-2)(x-3)(x+1)$; (vi.) $xy=2x+5y$; (vii.) $x^2y=(x-1)^2$; (viii.) $xy-3x-y+2=0$; (ix.) $xy^2=4(2-x)$; (x.) $4y+2xy-3x=0$.

Historical Note.

The idea of determining the position of a point by means of co-ordinates is due to Descartes, and occurred to him one morning when lying in bed. In 1637 he published at Leyden his *Discours*, an epoch-making work which for the first time brought to light the fundamental relation between geometry and algebra.

When a point moves along a straight line its co-ordinates x and y will vary, and Descartes showed that these co-ordinates were related by an algebraical expression of the form $ax+by+c=0$, where a and b remained constant for a given line. (The practice of denoting variables by the letters x and y, and constants by a, b, c, etc., is due also to Descartes.)

It was at once clear that a new and powerful weapon had been forged for the development of mathematical ideas. When a point moves along its locus the condition to be satisfied by its co-ordinates is given by an algebraic relation between x and y: so also to an expression of given algebraic form there correspond loci of some general type whose geometrical conditions are all of the same form: henceforth it was possible to solve geometrical problems by the rules of algebra and so avoid reference to complicated geometrical figures.

The co-ordinates x and y of a point are often called the Cartesian co-ordinates in honour of Descartes.

CHAPTER III

EQUATIONS OF TANGENTS AND NORMALS

Gradient of a Curve.

LET **P** (x, y) be any point on a straight line. If x be increased from **ON** to $\mathbf{ON_1}$ the increment in x is called δx, the ordinate corresponding to $\mathbf{ON_1}$ will be $\mathbf{N_1P_1}$, which is called $y+\delta y$ where $\mathbf{P_1R_1}$ is δy the increment in y.

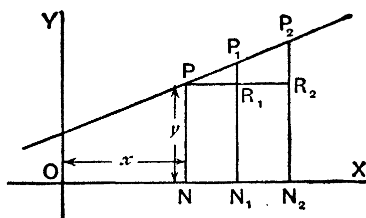


FIG. 36

The gradient of the line will be $\frac{\mathbf{P_1R_1}}{\mathbf{PR_1}}=\frac{\delta y}{\delta x}$.

If $\mathbf{NN_2}$ is taken to be δx, then $\mathbf{P_2R_2}$ will be δy, and the gradient is $\frac{\mathbf{P_2R_2}}{\mathbf{PR_2}}$, which has the same value as before, since $\frac{\mathbf{P_2R_2}}{\mathbf{PR_2}}=\frac{\mathbf{P_1R_1}}{\mathbf{PR_1}}$. The value obtained for the gradient is therefore independent of the size of δx and is also independent of the position of **P** for it will be the same at all points along the line.

Consider now the curve whose equation is $y=x^2$. See Fig. 37 (not drawn to scale).

Let **P** be any point (x, y) on the curve. If we take another point $\mathbf{Q_1}$ on the curve, the gradient of the chord $\mathbf{PQ_1}$ will be $\frac{\mathbf{Q_1R_1}}{\mathbf{PR_1}}$, and for another point $\mathbf{Q_2}$ the gradient of the

chord PQ_2 is $\frac{Q_2R_2}{PR_2}$; but $\frac{Q_2R_2}{PR_2}$ is not equal to $\frac{Q_1R_1}{PR_1}$, so that this gradient changes in value according to the nearness of

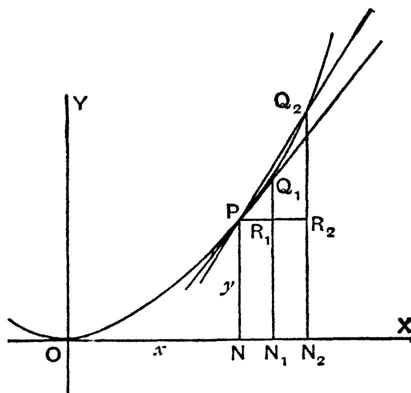


FIG. 37

Q to P. It will also depend on the position of the point P on the curve.

Suppose $ON=2$, then, since $y=x^2$ for any point on the curve, we have $PN=2^2=4$.

Now take an increment of ·1 in the abscissa so that $ON_1=2{\cdot}1$, then $Q_1N_1=(2{\cdot}1)^2=4{\cdot}41$.

. Q_1R_1 is therefore ·41, and PR_1 is ·1.

$$\therefore \frac{Q_1R_1}{PR_1}=4{\cdot}1$$

Now suppose NN_1 to be ·01, then $ON_1=2{\cdot}01$ and $Q_1N_1=(2{\cdot}01)^2=4{\cdot}0401$.

$$\therefore \frac{Q_1R_1}{PR_1}=\frac{{\cdot}0401}{{\cdot}01}=4{\cdot}01$$

If we make $NN_1={\cdot}001$ we get $\frac{Q_1R_1}{PR_1}=4{\cdot}001$, and the nearer we take Q_1 to P the smaller we make NN_1, and the

nearer does the gradient of the chord approach the value 4.

This value 4 to which the gradient of the chord more nearly approaches the nearer **Q** comes to **P**, is called the gradient of the curve at **P**.

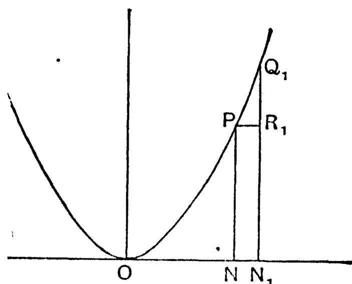


FIG. 38

If we take (x, y) as the co-ordinates of **P** and make an increment $\delta x = NN_1$ in x, then we make an increment R_1Q_1 in the ordinate which is called δy, so that $Q_1N_1 = y + \delta y$.

But since Q_1 is on the curve its co-ordinates must satisfy the relation $Q_1N_1 = ON_1^2$.

i.e. $$y + \delta y = (x + \delta x)^2 = x^2 + 2x\,\delta x + \overline{\delta x}|^2$$

But $y = x^2$ since **P** is on the curve.

$$\therefore \delta y = 2x\ \delta x + \overline{\delta x}|^2$$

$$\therefore \frac{\delta y}{\delta x} = 2x + \delta x$$

This is the gradient of the chord PQ_1 and as Q_1 comes nearer to **P**, δx approaches zero and the gradient approaches the value $2x$. $2x$ is called the gradient of the curve at the point **P**.

The value to which $\frac{\delta y}{\delta x}$ continually approaches as δx approaches zero, is called its limiting value, and is usually written $\frac{dy}{dx}$—*i.e.* $\frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{L}} \frac{\delta y}{\delta x}$ and in this case the value of the limit is $2x$, and at the point whose abscissa is 2 its value will be 4.

If the equation of the curve is $y = 3x^2$ the ordinate will be

three times the corresponding ordinate for the curve $y=x^2$.

$$\therefore \ Q_1N_1=3QN_1 \quad P_1N=3PN$$

$$\therefore \ Q_1R_1=3\ QR$$

and
$$\frac{Q_1R_1}{P_1R_1}=\frac{3QR}{PR}$$

Hence the gradient of $y=3x^2$ will be $3(2x)=6x$.

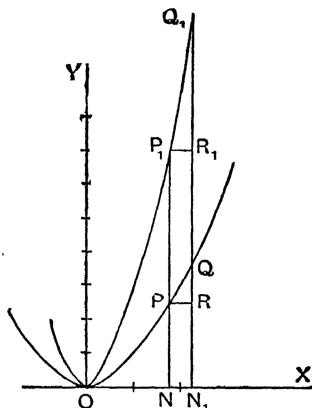


Fig. 39

This can be proved independently by noting that $y+\delta y=3(x+\delta x)^2$ ($Q_1N_1=3\ ON_1^2$) and by finding $\frac{\delta y}{\delta x}$ as before.

Examples. Verify the following results :—

Curve $y=x^4$, gradient $\frac{dy}{dx}=4x^3$

$y=3x^4$, gradient $\frac{dy}{dx}=12x^3$

$y=2$, gradient $\frac{dy}{dx}=0$

$y=c$ (a constant), gradient $\frac{dy}{dx}=0$

These curves are particular cases of the curve $y=ax^n$ for which $\frac{dy}{dx}=anx^{n-1}$.

It is shown in text-books on the Calculus that this result is true for all values of n whether integral, fractional, positive or negative.

Gradient of $y=\frac{1}{x}$.

As before we have $y+\delta y=\frac{1}{x+\delta x}$.

$$\therefore\ \delta y=\frac{1}{x+\delta x}-\frac{1}{x}=\frac{-\delta x}{x(x+\delta x)}$$

$$\therefore\ \frac{\delta y}{\delta x}=-\frac{1}{x(x+\delta x)}$$

$$\therefore\ \frac{dy}{dx}=-\frac{1}{x^2}=-x^{-2}$$

Here $y=x^{-1}$. If we apply the rule given on the previous page we have, since $n=-1$, $\frac{dy}{dx}=-1\ (x)^{-1-1}=-x^{-2}$.

Suppose the equation of the curve to be $y=ax^2+bx+c$. Then if (x, y) are the co-ordinates of P and $(x+\delta x, y+\delta y)$

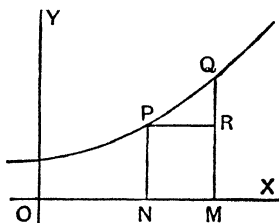


FIG. 40.

the co-ordinates of Q, we have since Q is on the curve,

$$y+\delta y=a(x+\delta x)^2+b(x+\delta x)+c$$

$$=ax^2+2ax\ \delta x+a\overline{\delta x}|^2+bx+b\delta x+c$$

But $$y=ax^2+bx+c$$

$$\therefore\ \delta y=2ax\delta x+a\overline{\delta x}|^2+b\delta x$$

$$\therefore\ \frac{\delta y}{\delta x}=2ax+a\delta x+b$$

and
$$\frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{L}} \frac{\delta y}{\delta x} = 2ax + b$$

We see then that for an equation consisting of several terms the rule may be applied to each term separately.

Gradient of the Circle $x^2+y^2=a^2$.

Let (x, y) be the co-ordinates of P $(x+\delta x, y+\delta y)$ the co-ordinates of Q.

(If δx is positive we see from a figure that δy will be

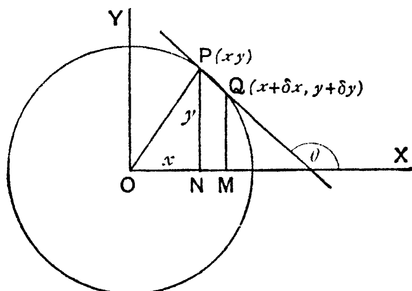


FIG. 41

negative, but we shall work independently of the figure and δy will carry its own sign.)

Since Q is on the curve we have

$$(x+\delta x)^2+(y+\delta y)^2=a^2$$

i.e.
$$x^2+2x\ \delta x+\overline{\delta x}|^2+y^2+2y\ \delta y+\overline{\delta y}|^2=a^2$$

$$\therefore\ 2x\ \delta x+\overline{\delta x}|^2+2y\ \delta y+\overline{\delta y}|^2=0$$

$$\therefore\ 2x+\delta x+2y\frac{\delta y}{\delta x}+\delta y\ \frac{\delta y}{\delta x}=0$$

As δx approaches zero, δy also approaches zero, and $\frac{\delta y}{\delta x}$ approaches its limiting value, which we denote by $\frac{dy}{dx}$

$$\therefore\ 2x+2y\ \frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

$\therefore$ at a point on the curve whose co-ordinates are $(x_1 y_1)$ the gradient is $-\frac{x_1}{y_1}$.

The result shows that $\tan \theta = -\frac{x}{y} = -\cot$ PON (Fig. 41).

Hence $\theta = (90° + \text{PON})$—*i.e.* the tangent at P is perpendicular to OP.

The student familiar with the differential notation will at once obtain the gradient from the equation $x^2 + y^2 = a^2$.

We have $2x\ dx + 2y\ dy = 0$.

$$\therefore \frac{dy}{dx} = -\frac{x}{y} \text{ which equals } -\frac{x_1}{y_1} \text{ at the point } x_1 y_1$$

Equation of the Tangent to a Curve.

To find the equation of the tangent to the curve $y = x^2$ at the point (1, 1).

The gradient of the tangent at the point whose co-ordinates are x_1 and y_1 is given by $\frac{dy}{dx} = 2x_1$

$\therefore$ at the point (1, 1) the gradient is 2

Since the tangent is a line through the point (1, 1) whose gradient is 2 the equation is

$$y - y_1 = m(x - x_1)$$

i.e.
$$y - 1 = 2(x - 1)$$

EXAMPLE. Find the gradient of the tangent to $y^2 = x$ at the point $x_1 y_1$.

Let P (x, y) be any point on the curve and Q $(x + \delta x, y + \delta y)$ another point on the curve near P.

Then $(y + \delta y)^2 = (x + \delta x)$.

$$\therefore y^2 + 2y\ \delta y + (\delta y)^2 = x + \delta x$$
$$\therefore 2y\ \delta y + (\delta y)^2 = \delta x \text{ since } y^2 = x$$
$$\therefore 2y \frac{\delta y}{\delta x} + \frac{\delta y}{\delta x} \cdot \delta y = 1$$

As **Q** approaches **P**, $\frac{\delta y}{\delta x}$ approaches $\frac{dy}{dx}$ the gradient of the tangent, and δy approaches 0.

$$\therefore 2y\frac{dy}{dx}=1 \qquad \therefore \frac{dy}{dx}=\frac{1}{2y}$$

$\therefore$ at x_1y_1 the gradient of the tangent is $\frac{1}{2y_1}$

This result could be obtained by writing the equation of the curve in the form $y=x^{\frac{1}{2}}$, then $\frac{dy}{dx}=\frac{1x^{-\frac{1}{2}}}{2}=\frac{1}{2\sqrt{x}}=\frac{1}{2y}$ (since $y=x^{\frac{1}{2}}$ for any point on the curve).

The value of the gradient $\frac{dy}{dx}=\frac{1}{2y}$ depends of course on y, and varies as we move along the curve. At a given point (x_1y_1) the gradient will be $\frac{1}{2y_1}$. A common error is to write $\frac{1}{2y}$ as the gradient when writing the equation of the tangent at x_1y_1, but as this leads to a quadratic expression in y for the equation of the tangent the error is at once detected.

EXAMPLES VI

1. Find the equations of the tangents to the following curves at the point **P**:—(i.) $y=3x^2$, **P** (1, 3); (ii.) $y=x^2+2x+1$, **P** (0,1); (iii.) $y=\frac{1}{x^2}$, **P** (1, 1); (iv.) $x^2+y^2=5$, **P** (2,1); (v.) $x^2+2x+y^2=0$, **P** (−1, 1); (vi.) $x^2-9y^2=81$, **P** (15, −4); (vii.) $x^2+2xy=4$, **P** $(x_1\ y_1)$; (viii.) $16y=x^4$, **P** (x_1y_1).

2. Show that the point $(at^2, 2at)$ lies on the parabola $y^2=4ax$. Find the equation of the tangent to the parabola $y^2=4ax$ at the point $(at^2, 2at)$.

3. Find the equation of the tangent to $xy=c$ at the point (x_1y_1). If this tangent cuts the axes at **P** and **Q** prove that the area of the triangle **POQ** is constant for any position of the tangent.

4. Find the angle in degrees which the tangent at (1, 1) to the curve $y=x^2$ makes with the **X** axis. Calculate also the gradient of the chord of this curve joining the points given by $x=1\cdot0$ and $x=1\cdot1$.

5. If the tangent to $y^2=4x$ at **P** (4, 4) meets the **X** axis at **T** find the co-ordinates of **T**. If **S** be the point 1, 0 find the length of **PS** and prove that it equals **ST**. Prove the same result if the co-ordinates of **P** are x_1y_1.

6. Find the equation of the tangent to $y^2=6x$ at x_1y_1. Find the co-ordinates of the point on the curve at which the tangent is parallel to $x=4y$.

7. Find the gradient and the equation of the tangent to $x^2-y^2=4$ at the point $\left(\frac{5}{2}, \frac{3}{2}\right)$.

If **P** (x_1y_1) is any point on the curve prove that the tangent at **P** makes the same angle with the **X** axis that **OP** makes with the **Y** axis. If this tangent meets the lines $y=x$ and $y=-x$ at **Q**, **R** respectively, find the co-ordinates of **Q** and **R** and the area of the triangle **QOR**.

8. Find the equation of the tangent to $y^2=7x$ which is parallel to the line $4y-x+3=0$. Find also its point of contact.

9. Find the co-ordinates of the two points where the tangent at x_1y_1 to $x^2+y^2=9$ cuts the two axes.

10. Find the co-ordinates of the point of intersection of the tangents to $y=x^2$ at the points $(-1,1)$ and $(2,4)$.

11. Show that at the point of intersection of the curves (i.) $xy=c$ and (ii.) $y^2=4ax$ the gradient of (i.) added to twice the gradient of (ii.) equals zero.

12. Find the equation of the tangent at (1, 1) to the curve $y=x^3$. Find the co-ordinates of the points where the tangent cuts the axes.

13. Find the gradient of the tangent at the point where $x=16a$ on the curve $x+y=a^{\frac{1}{4}}x^{\frac{3}{4}}$. Find where this tangent meets the line $x+y=0$.

14. Find the equation of the tangent to the curve $y=3x^4-4x^3-12x^2+5$ at the point where $x=3$. Find also the co-ordinates of the points at which the tangent is parallel to **OX**.

The Angle between Two Lines. Condition of Perpendicularity.

Let the equations of the two lines be $x+2y-6=0$ and $3x-y-2=0$.

Writing them in the form $y=mx+b$ we have $y=-\frac{1}{2}x+3$ and $y=3x-2$.

If θ_1, θ_2 are the angles they make with the axis OX, then $\tan\theta_1=-\frac{1}{2}$; $\tan\theta_2=3$.

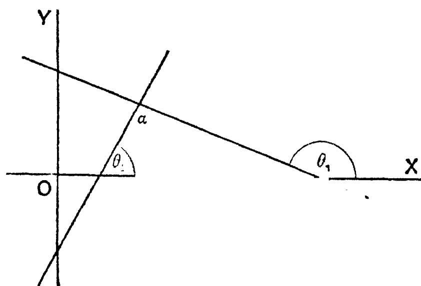


FIG. 42

If α is the angle between the lines, $\alpha=\theta_1-\theta_2$.

$$\therefore \tan\alpha=\tan(\theta_1-\theta_2)=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}=\frac{-\frac{1}{2}-3}{1+\left(-\frac{1}{2}\right)(3)}=7$$

If the equations of the lines are $y=m_1x+b_1$ and $y=m_2x+b_2$ then $\tan\theta_1=m_1$ and $\tan\theta_2=m_2$.

Also $\tan\alpha=\tan(\theta_1-\theta_2)=\frac{m_1-m_2}{1+m_1m_2}$.

If the lines are parallel $\theta_1=\theta_2$ $\therefore m_1=m_2$.

If the lines are at right angles $\alpha=90°$ and $\tan(\theta_1-\theta_2)=\infty$.

$$\therefore 1+m_1m_2=0$$

$$\text{or } m_2=-\frac{1}{m_1}$$

This condition for perpendicularity may also be obtained thus:

If $\alpha=90°$ then $\theta_1=90°+\theta_2$.

$$\therefore \tan\theta_1=\tan(90°+\theta_2)=-\cot\theta_2$$

$$i.e.\ m_1=-\frac{1}{m_2}$$

Equation of the Normal to a Curve.

The normal is a line at right angles to a tangent through its point of contact.

The normal to the parabola $y=x^2$ at the point (1, 1) is a

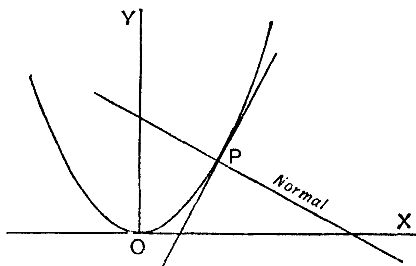


FIG. 43

line at right angles to the tangent, $y-1=2(x-1)$ (see p. 36), through the point (1, 1).

Since the gradient of the tangent is 2 the gradient of the normal is $-\frac{1}{2}$, $m_1=-\frac{1}{m_2}$ being the condition for perpendicularity.

The normal passes through the point (1, 1) $\therefore$ its equation is $y-1=-\frac{1}{2}(x-1)$.

EXAMPLE. Find the equation of the line through the point (3, 2) which is at right angles to $2x-3y+4=0$.

The gradient of the given line is $\frac{2}{3}$ $\therefore$ the gradient m of a

line perpendicular to it is $-\frac{3}{2}$. Since the line has to pass through (3, 2) its equation is $y-2=m(x-3)$

or $$y-2=-\frac{3}{2}(x-3)$$

i.e. $$3x+2y-13=0$$

Length of the Perpendicular from (x_1y_1) on $Ax+By+C=0$.

Let AB be the line $Ax+By+C=0$ and let the angle which PN makes with OX be θ where PN is the perpendicular (p) from (x_1y_1) on AB.

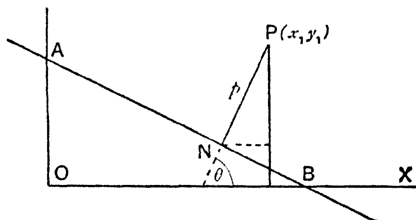


FIG. 44

The co-ordinates of N are $x_1-p\cos\theta$, and $y_1-p\sin\theta$. Since this point is on AB we have $A(x_1-p\cos\theta)+B(y_1-p\sin\theta)+C=0$.

$$\therefore p=\frac{Ax_1+By_1+C}{A\cos\theta+B\sin\theta}$$

Now the gradient of AB is $-\frac{A}{B}$

$$\therefore \text{the gradient of PN is } \frac{B}{A}=\tan\theta$$

$$\therefore \sin\theta=\frac{B}{\sqrt{A^2+B^2}} \text{ and } \cos\theta=\frac{A}{\sqrt{A^2+B^2}}$$

$$\therefore p=\frac{Ax_1+By_1+C}{\frac{A^2}{\sqrt{A^2+B^2}}+\frac{B^2}{\sqrt{A^2+B^2}}}=\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}$$

It will be shown in Chapter IV. that a significance is attached to the sign of this expression.

EXAMPLES VII

1. Find the tangent of the angle between the following pairs of lines:—(i.) $2x-y+3=0$, $x+4y-2=0$; (ii.) $x+\frac{5}{3}y+\frac{1}{2}=0$, $\frac{x}{2}-\frac{1}{4}y=3$; (iii.) $x-2y=3$, $y+2x=4$.

2. Find the angles of the triangles whose angular points are (i.) (1, 3), (0, 2), (−1, 2); (ii.) (0, 0), (3, 3), (5, 1).

3. Find the equation of the normal to $8y=x^2$ at the point on the curve whose abscissa is 4. Where does the normal cut the Y axis?

4. A line is drawn through the point P (3, 2) perpendicular to the line joining P to the origin. Find its equation and prove that the point (1, 5) lies on it.

5. Find the equation of a line through (4, 3) parallel to $2x+3y=5$.

6. A straight line is drawn through the mid-points of AB, AC where ABC is the triangle, A (1, 1), B (3, 4), C (5, −2). Find its equation and show that it is parallel to BC.

7. Find the equation of the tangent at x_1y_1 to the parabola $y^2=4ax$. Find also the equation of a line from $(a, 0)$ perpendicular to the tangent. Prove that they meet on the line $x=0$.

8. Prove that for all values of k the line $2x-3y+k=0$ is perpendicular to $3x+2y+1=0$, and the line $bx-ay+k=0$ perpendicular to $ax+by+c=0$.

9. Find the lengths of the perpendiculars from the given point A to the given line when (i.) A (0, 0), $4x+3y-4=0$; (ii.) A (−2, 1), $4x-3y+1=0$; (iii.) A (4, 5), $8x+15y=4$; (iv.) A (2, −3), $3x+2y=6$.

10. Find the equations of lines through the origin perpendicular to (i.) $\frac{x}{a}+\frac{y}{b}=1$; (ii.) $y=mx+b$; (iii.) $x\cos\alpha+y\sin\alpha=p$.

11. Prove that A (1, 3), B (4, 4), C (5, 1), D (2, 0) are the corners of a square. Find the equations of its diagonals and show that they are at right angles to each other.

12. **A** (a, o), **B** $(-a, o)$ are two fixed points. **P** is the point $(x_1 y_1)$, find the equations of the lines **PA**, **PB**, and if they are perpendicular to one another prove $x_1^2+y_1^2=a^2$.

13. **CAB** is a right-angled triangle. A line drawn perpendicular to the hypotenuse **AB**, meets **CB** at **P** and **CA** at **Q**. Prove analytically that **BQ** is perpendicular to **AP**.

14. **B** is the point (o, b). Write down the equation of a line through **B** whose gradient is m. Write down also the equation of a line through **B**^{1} $(o, -b)$ perpendicular to this line. If (x, y) are the co-ordinates of the point where these lines intersect prove $x^2+y^2=b^2$.

15. Find the angles of the triangle formed by the lines $3x+2y-4=0$, $x-3y+6=0$, $4x-3y-10=0$.

16. Find the equations of the lines drawn through the vertices of the triangle **ABC** which are perpendicular to the opposite sides, given **A** $(-3, 2)$, **B** $(3, -2)$, **C** $(0, -1)$. Show that they meet in a point and find its co-ordinates.

17. Find the co-ordinates of the centre of the circumscribing circle of the triangle whose vertices are **A** $(0, 4)$, **B** $(3, 5)$, **C** $(5, 1)$.

18. Find the equations of the two straight lines through $(1, 3)$, each of which cuts off along **OX** a length double that which it cuts off along **OY**. Find the angle between these two lines.

19. Find the equations of two lines through $(3, 2)$ forming two sides of a square, of which $4x+7y-12=0$ is one diagonal.

20. Find the lengths of the perpendiculars from the vertices of the triangle **A** $(-2, 1)$, **B** $(1, 4)$, **C** $(3, -1)$ to the opposite sides.

21. Find the points of intersection of the parabola $y^2=16x$ and the line $y=2x-6$. Find the equations of the tangents to the parabola at these points and the angles between the line and the tangents.

22. Find the points of intersection of $y^2=16-8x$ and $y^2=2x+1$. Find the tangents to both curves at these points and hence show that the curves cut at right angles.

23. Find the equation of the tangent at a point **P** whose abscissa is 8 on the curve $y = 8x$. If this tangent meets the line $x = -2$ at **Z**, find the gradients of the lines joining **S** (4, 0) to **P** and to **Z**. Show that these lines are perpendicular.

24. Find the angle between the tangents to the curve $y^2 = 5x$ at the points where it is cut by the line $y = 2x - 8$. Find also the co-ordinates of the point of intersection of the tangents.

CHAPTER IV

STRAIGHT LINE (*continued*)

(This chapter may be omitted on first reading)

Intercept Form of the Equation to a Straight Line.

THE position of a straight line is fixed when the intercepts it makes on the axes are known.

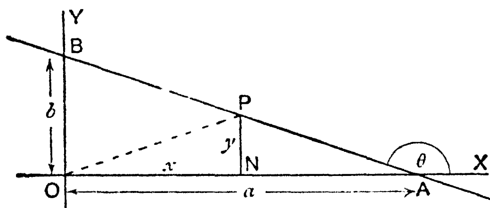


FIG. 45

Let the intercept on the axis **OX** be a, and on the axis **OY** be b.

Note that we draw the figure for the case in which a and b are both positive.

Let **P** (x, y) be any point on the line. Then from similar triangles

$$\frac{BO}{OA}=\frac{PN}{NA} \qquad i.e.\ \frac{b}{a}=\frac{y}{a-x}$$

or $\quad ab-bx=ay \qquad i.e.\ bx+ay=ab$

$$\therefore \frac{x}{a}+\frac{y}{b}=1$$

Alternative Method.

Join **OP**. Then area of triangle **BOA**$=\overline{\textbf{BOP}}+\overline{\textbf{POA}}$.

i.e.
$$\frac{1}{2}ab=\frac{1}{2}bx+\frac{1}{2}ay$$

or
$$1=\frac{x}{a}+\frac{y}{b}$$

Writing this in the form $y=mx+b$ we have $y=-\frac{b}{a}x+b$.

Hence $m=\tan\theta=-\frac{b}{a}$, which is obvious from the figure.

To write $y=mx+b$ in the intercept form we have $-mx+y=b$.

or
$$\frac{-mx}{b}+\frac{y}{b}=1$$

i.e.
$$\frac{x}{\left(\frac{-b}{m}\right)}+\frac{y}{b}=1$$

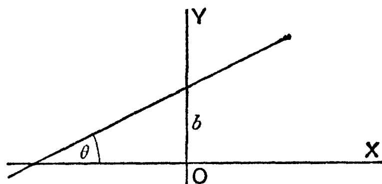


FIG. 46

Hence the intercepts are $\frac{-b}{m}$, *i.e.* $-b\cot\theta$ and b, as is clear from Fig. 46.

Perpendicular Form.

If the position and length of the perpendicular drawn from the origin to a line are fixed, then the position of the line is fixed.

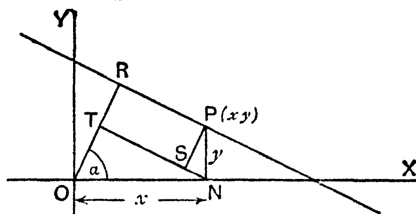


FIG. 47

Let the perpendicular from the origin be p and let the angle it makes with the positive direction of the X axis be α.

Let **P** (x, y) be any point on the line. (*N.B.*—Do not take **P** at **R**, which is not *any* point.)

Draw **NT** perpendicular to **OR**. Draw **SP** parallel to **OR**. Then $\widehat{SNP}=\alpha$.

$$\therefore \text{OT}=x\cos\alpha;\ \text{TR}=\text{SP}=y\sin\alpha, \text{ but OR}=p$$

$$\therefore x\cos\alpha + y\sin\alpha=p$$

To write this in the form $y=mx+b$ we have $y=-x\cot\alpha + p\operatorname{cosec}\alpha$.

$$\therefore m=\tan\theta=-\cot\alpha \text{ and } b=p\operatorname{cosec}\alpha$$

To write $x\cos\alpha+y\sin\alpha=p$ in the intercept form we have

$$\frac{x\cos\alpha}{p}+\frac{y\sin\alpha}{p}=1$$

or

$$\frac{x}{p\sec\alpha}+\frac{y}{p\operatorname{cosec}\alpha}=1$$

Hence the intercepts are $p\sec\alpha$ and $p\operatorname{cosec}\alpha$.

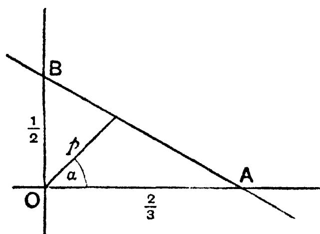


FIG. 48

To write the equation $3x+4y=2$ in the perpendicular form.

The intercepts on the axes are $\frac{2}{3}$ and $\frac{1}{2}$.

$$\therefore \text{AB}=\sqrt{\frac{4}{9}+\frac{1}{4}}=\frac{5}{6}$$

Also $$\hat{OBA}=\alpha$$

$$\therefore \cos\alpha=\frac{\frac{1}{2}}{\frac{5}{6}}=\frac{3}{5} \text{ and } \sin\alpha=\frac{4}{5}$$

$$p=\frac{1}{2}\sin \hat{OBA}=\frac{2}{5}$$

$$\therefore \text{ the equation is } \frac{3}{5}x+\frac{4}{5}y=\frac{2}{5}$$

Note that this is the original equation with both sides divided by 5.

Perpendicular Form of the General Equation.

To write the general equation $ax+by+c=0$ in the form $x\cos\alpha+y\sin\alpha=p$.

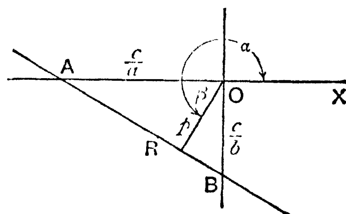


Fig. 49

The intercepts are $-\frac{c}{a}$ and $-\frac{c}{b}$.

Hence $$AB=\sqrt{\frac{c^2}{a^2}+\frac{c^2}{b^2}}=\frac{c}{ab}\sqrt{a^2+b^2}$$

Also α the angle the perpendicular OR makes with the positive direction of the X axis is XOR.

Let AOR be β then $\hat{ABO}=\beta$.

$$\therefore \cos\beta=\frac{\frac{c}{b}}{AB}=\frac{a}{\sqrt{a^2+b^2}} \text{ and } \sin\beta=\frac{b}{\sqrt{a^2+b^2}}$$

Also $$p=\frac{c}{b}\sin\beta=\frac{c}{\sqrt{a^2+b^2}}$$

Now $\alpha=180°+\beta$ $\therefore$ $\cos\alpha=-\cos\beta$ and $\sin\alpha=-\sin\beta$.

$$\therefore \cos\alpha=-\frac{a}{\sqrt{a^2+b^2}} \text{ and } \sin\alpha=-\frac{b}{\sqrt{a^2+b^2}}$$

Hence the equation is $-\frac{ax}{\sqrt{a^2+b^2}}-\frac{by}{\sqrt{a^2+b^2}}=\frac{c}{\sqrt{a^2+b^2}}$

If will be noted that this is the original equation with both sides divided by $\sqrt{a^2+b^2}$, and that the proper signs are given to $\cos\alpha$ and $\sin\alpha$ if c is taken to the right-hand side of the equation and made positive.

N.B.—The equation $x\cos 45°-y\sin 45°=-2$ represents a line such that the angle between the perpendicular from the origin and the **X** axis is 135°, *not* 45°, as may be seen by writing it with p positive, thus: $-x\cos 45°+y\sin 45°=+2$ or $x\cos 135°+y\sin 135°=2$.

Alternative Method.

An equation can only be written in a new form by multiplying throughout by some constant.

Let the required multiplier be k, then $kax+kby+kc=0$. *i. e.* $-kax-kby=kc$ is to be the same as $x\cos\alpha+y\sin\alpha=p$.

$$\therefore \cos\alpha=-ka \text{ and } \sin\alpha=-kb$$

but $\cos^2\alpha+\sin^2\alpha=1$ $\therefore$ $k^2a^2+k^2b^2=1$

$$\therefore k=\frac{1}{\sqrt{a^2+b^2}}$$

Hence the equation becomes

$$-\frac{ax}{\sqrt{a^2+b^2}}-\frac{by}{\sqrt{a^2+b^2}}=\frac{c}{\sqrt{a^2+b^2}}$$

EXAMPLES VIII

1. Find the equation of the line making intercepts of **2** and **—3** on the axes of **X** and **Y** respectively.

2. Find the equation of the line passing through the point **(2,4)** and making equal intercepts on the axes (i.) both positive: (ii.) one negative, the other positive.

3. Write the following equations in the perpendicular form :—
(i.) $y=3x+2$; (ii.) $\frac{x}{2}+\frac{y}{3}=1$; (iii.) $4x-3y+1=0$.

4. Show from a diagram that the length of the perpendicular from x_1y_1 to $3x+2=0$ is $\frac{3x_1+2}{3}$ and the perpendicular from x_1y_1 to $4x-3$ is $\frac{4x_1-3}{4}$.

5. Write the equation $x \cos 45°+y \sin 45°=\sqrt{2}$ in the intercept form.

6. What is the length of the perpendicular from the origin to the lines (i.) $x \cos 30°+y \sin 30°=2$; (ii.) $3x-4y=10$?

7. Find the equation of the line through the origin perpendicular to $x \cos 30°+y \sin 30°=2$.

8. Find the equation of the line through (4, 5) perpendicular to $x \cos 60°+y \sin 60°=3$. If the lines intersect at **P** find the co-ordinates of **P**.

9. Find the equation of a line in the perpendicular form which goes through (4, 3) and is parallel to $x \cos 60°+y \sin 60°=2$. How far apart are these lines?

10. Find the equation of a straight line which makes a negative intercept of 3 units on the **X** axis and passes through the point (1, 2·5).

11. A straight line cuts the axis of **X** at **A** and the axis of **Y** at **B** where **OA**$=4$, **OB**$=6$. If **P** is any point (x, y) on **AB**, find the relation between x and y. If (**XY**) are the co-ordinates of **M** the mid-point of **OP**, show that $\frac{X}{2}+\frac{Y}{3}=1$.

12. A straight line is drawn through the point 3, 2 which cuts **OX** at **A** and **OY** at B. The rectangle **OAPB** is completed and the co-ordinates of **P** are taken to be **X**, **Y**. Write down the equation of **AB** and show that $3Y+2X=XY$.

13. A straight line passes through the point 4, 4 and cuts the axes at **A** and **B**. Prove that for all positions of the line $\frac{1}{OA}+\frac{1}{OB}=\frac{1}{4}$.

14. From the point **P** (3, 4) perpendiculars **PM**, **PN** are drawn to the axes. Write the equation of the line **NM** in the perpendicular form and hence find the length of the perpendicular from **O** to **NM**.

15. Find the angle between **OX** and the perpendicular from **O** on the line $3x+4y+2=0$.

16. Find the angle between the two lines $3x+4y+5=0$ and $4x+3y+10=0$ by finding the angles between **OX** and the perpendiculars from the origin on these lines.

Length of the Perpendicular from a Point to a Line.

When the equation of a straight line is written in the form $x \cos \alpha + y \sin \alpha = p$ the length of the perpendicular **PR** from a point **P** (4, 3) is easily found.

Suppose the equation of the line to be

$x \cos 45° + y \sin 45° = 2$.

Through **P** draw a line parallel to the given line and produce the perpendicular from the origin to meet this line at **T**.

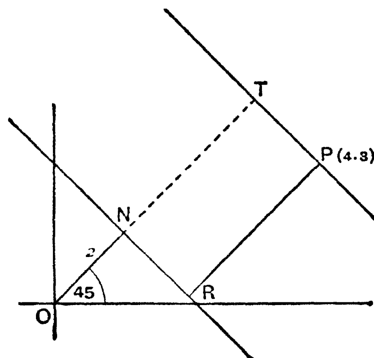


FIG. 50

The perpendicular from the origin to the line **PT** is 2+**NT** and the angle this perpendicular makes with **OX** is 45°. Therefore the equation of the line **PT** is

$$x \cos 45° + y \sin 45° = 2 + \mathbf{NT}$$

Now the point (4, 3) is on this line, so that

$$4 \cos 45° + 3 \sin 45° = 2 + \mathbf{NT}$$

$$\therefore \mathbf{PR} = \mathbf{NT} = 4 \cos 45° + 3 \sin 45° - 2$$

If the equation of the original line were $\frac{4}{5}x + \frac{3}{5}y = 2$ then **PR** would be $4(\frac{4}{5}) + 3(\frac{3}{5}) - 2 = 3$.

If the co-ordinates of **P** are $x_1 y_1$ then since $x_1 y_1$ is on the

line **PT**, whose equation is $x \cos \alpha + y \sin \alpha = p + \mathbf{PR}$, we have the relation $x_1 \cos \alpha + y_1 \sin \alpha = p + \mathbf{PR}$.

$$\therefore \mathbf{PR} = x_1 \cos \alpha + y_1 \sin \alpha - p$$

Graphically.

The length of the perpendicular from **A** $(x_1 y_1)$ on the line whose equation is $x \cos \alpha + y \sin \alpha = p$ may be obtained immediately by projection. Let **P** (**AT**) be the required per-

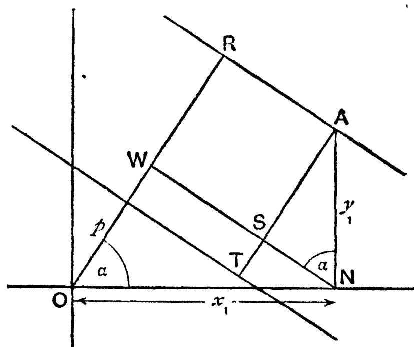


FIG. 51

pendicular. Projecting x_1 and y_1 on **OR** we have $\mathbf{OW} + \mathbf{WR} = p + \mathbf{P}$, but $\mathbf{OW} = x_1 \cos \alpha$ and $\mathbf{WR} = \mathbf{AS} = y_1 \sin \alpha$.

$$\therefore x_1 \cos \alpha + y_1 \sin \alpha - p = \mathbf{P}$$

EXAMPLE. To find the length of the perpendicular from the point (4, 3) on the line $12x - 5y = 12$.

Writing the equation of the line in the perpendicular form we have:

$$\frac{12}{13}x - \frac{5}{13}y = \frac{12}{13} \text{ since } \sqrt{a^2+b^2} = \sqrt{12^2+(-5)^2} = 13$$

The line through (4, 3) parallel to the given line will have for its equation:

$$\frac{12}{13}x - \frac{5}{13}y = \frac{12}{13} + \mathbf{P}$$

where **P** is the required perpendicular from (4, 3).

. Since the point (4, 3) is on this line we have:

$$\frac{12}{13}\cdot4-\frac{5}{13}\cdot3=\frac{12}{13}+P$$

$$\therefore\ P=\frac{21}{13}$$

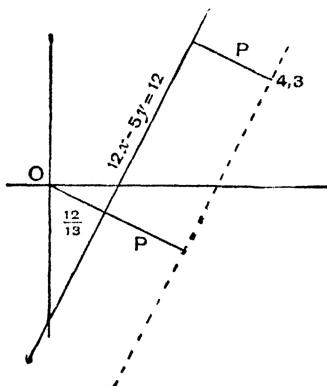


FIG. 52

Length of the Perpendicular from x_1y_1 to the Line $ax+by+c=o$.

The method is the same as in the former case provided the equation of the given line is written in the form $x\cos\alpha+y\sin\alpha=p$.

The given line is $-\frac{ax}{\sqrt{a^2+b^2}}-\frac{by}{\sqrt{a^2+b^2}}=\frac{c}{\sqrt{a^2+b^2}}$

$\therefore$ the length of the perpendicular from x_1y_1 is

$$-\frac{ax_1}{\sqrt{a^2+b^2}}-\frac{by_1}{\sqrt{a^2+b^2}}-\frac{c}{\sqrt{a^2+b^2}}$$

i.e. $\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$ if we neglect the sign.

Sign of the Perpendicular.

In trigonometry the line describing an angle is taken to be positive, so in co-ordinate geometry the perpendicular **ON** from the origin on a line is always positive.

If x_1y_1 is on the line $ax+by+c=0$ then $ax_1+by_1+c=0$ and the perpendicular from x_1y_1 on the line is zero. We should therefore expect the perpendicular to change sign as x_1y_1 passes from one side of a line to the other.

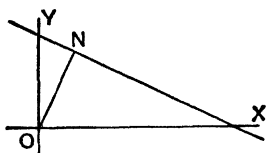


FIG 53

When a point is on the same side of a line as the origin the perpendiculars from the point and the origin have the same sign.

When the point and the origin are on opposite sides of the line the perpendiculars are of opposite sign.

In many cases the *length* only of the perpendicular is required, and then we can ignore the question of sign.

When perpendiculars on to one line are concerned it is generally only necessary to determine whether or not the perpendiculars are on the origin side of the line.

In some cases, however, it is necessary to give the perpendicular its correct sign, and this is determined by writing the expression for its length in such a way as to make the perpendicular from the origin positive.

The length of the perpendicular from (2, 3) to the line $x \cos 30°+y \sin 30°=4$ is by the formula $2 \cos 30°+3 \sin 30° -4= -{\cdot}77$, whereas the perpendicular from the origin is $0+0-4=-4$. The point (2, 3) is therefore on the same side of the line as the origin, since both perpendiculars are negative. We have agreed, however, that perpendiculars from the origin shall be positive, hence the length of the perpendicular from (0, 0) is $-(0+0-4)=4$ and from (2, 3) it is $-(2 \cos 30+3 \sin 30-4)=+{\cdot}77$.

As another example, suppose we require to express the fact that the perpendiculars from P (x_1y_1) on the lines $3x+4y=12$ and $12x+5y+24=0$ are equal and that P is on the bisector of the acute angle between the lines.

A figure will show that P and the origin are on the same

side of the line $3x+4y=12$; P is also on the origin side of $12x+5y+24=0$, but the perpendiculars required have different signs—thus:

The perpendicular from x_1y_1 on $3x+4y-12=0$ is $-\left(\frac{3x_1+4y_1-12}{5}\right)$ since the perpendicular from (0, 0) must be positive $\left(\frac{12}{5}\right)$.

The perpendicular from x_1y_1 on $12x+5y+24=0$ is $+\left(\frac{12x_1+5y_1+24}{13}\right)$, in order to make the perpendicular from (0, 0) positive $\left(\frac{24}{13}\right)$.

Hence the relation is

$$-\left(\frac{3x_1+4y_1-12}{5}\right)=+\left(\frac{12x_1+5y_1+24}{13}\right)$$

EXAMPLES IX

1. Find the lengths of the perpendiculars from (i.) (0, 0) to $x+2y+3=0$; (ii.) (2, -1) to $x+3y+1=0$; (iii.) (5, 1) to $3x-4y+2=0$; (iv.) (x_1y_1) to $4x+3y+1=0$; (v.) (1, 1) to $2x\cos 30°+3y\sin 30°=4$; (vi.) (2, 1) to $x\sin 60°+y\cos 60°=2$.

2. Find the length of the perpendiculars from (i.) (2, 3) to $\frac{x}{3}+\frac{y}{4}=1$; (ii.) (1, 4) to $\frac{x}{12}-\frac{y}{5}=2$; (iii.) (a, o) to $y=mx+\frac{a}{m}$.

3. Find the distance between the parallel lines $y=2x-4$ and $y=2x+3$ by finding the lengths of the perpendiculars to them from the origin. Check from a figure.

4. Find the lengths of the perpendiculars from the vertices to the opposite sides of the triangle whose vertices are (0, 1), (1, 3), (4, -2).

5. Find the length of the perpendicular from (2, 0) to the tangent drawn to the curve $y^2=8x$ at the point P on it whose ordinate is 4.

6. Find the length of the perpendicular from (4, 2) to the tangent to the curve $xy=2$ at the point (1, 2).

7. Find the distance apart of the parallel lines $2x+y=6$. $6x+3y=8$.

8. Find the length of the perpendicular from **P** (**X**, **Y**) on $3x-4y=5$. If this length equals the distance of **P** from the point (4, 0) find the relation between **X** and **Y**.

9. Determine whether the given point, **P**, is on the same or the opposite side of the given line as the origin. (i.) **P** (2, 3), $2x+3y-5=0$; (ii.) **P** (2, 2), $2x+5y=10$; (iii.) **P** (2, -1), $2x+5y=10$; (iv.) **P** (0, 9), $8x-3y+12=0$.

10. Find the perpendicular distances of the points (4, 1) and (-1, -1) from the line $2x+y-3=0$. Show the results on a graph.

11. From the point **P** (4, 3) perpendiculars **PA**, **PB** are drawn to the axes. Find the equation of **AB** and length of the perpendicular from **P** to the line **AB**.

12. The point **A** (o, b) is joined to the points **B** (a, o) and **C** ($-a$, o). Write down the equations of **AB** and **AC** and find the length of the perpendiculars **PM**, **PN** drawn to **AB** and **AC** from any point **P** (**X**, 0) on **BC**. Hence show that **PM** +**PN**=constant.

13. A line cuts the axis at **A** and **B** where **OA**=3, **OB**=4. The point **C** (2, 5) is joined to **A** and **B**. Find the area of the triangle **ACB**.

14. A point moves at a distance of 2 units from the line $5x+12y=13$. Write down the equations of its paths.

15. Find the perpendicular distance between the parallel lines $3x+4y+5=0$ and $6x+8y-11=0$.

16. Write down the equation of the tangent at **P** (x_1y_1) to the parabola $y^2=4ax$ and find the co-ordinates of **T** where this tangent cuts the **X** axis. Find the length of the perpendicular **SY** from **S** (a, o) to the tangent, and prove that $\mathbf{SY}^2=a\ \mathbf{ST}$.

17. A tangent is drawn to the circle $x^2+y^2=a^2$ at the point **P** (x_1y_1). Find the lengths of the perpendiculars **AN**, $\mathbf{A^1N^1}$ drawn to this tangent from the points **A** (b, o), $\mathbf{A^1}$ ($-b$, o). Show that $\mathbf{AN}+\mathbf{A^1N^1}=2a$.

18. Find the condition that the line $xx_1+yy_1=a^2$ should be at a distance a from the origin.

19. Find the equation of a line through the point (3, 4) whose distance from the origin is 5.

• 20. A line through the point (1, 1) cuts the axes at **A** and **B**. Prove that the perpendicular from **O** to **AB** is $\frac{OA+OB}{AB}$.

Equation of the Line bisecting the Angle between the Lines $3x-4y=6$ **and** $5x+12y=24$.

Let $P\ (X,\ Y)$ be any point on the required bisector, then

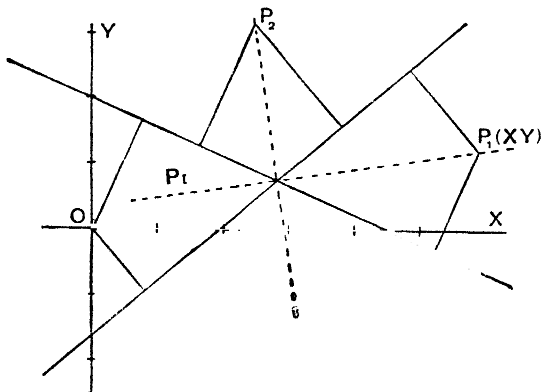


Fig. 54

the perpendiculars from **P** to the two lines must be of equal length—*i.e.*

$$\frac{3X-4Y-6}{\sqrt{9+16}}=\pm\frac{5X+12Y-24}{\sqrt{25+144}} \quad \text{(i.)}$$

If **P** is at P_1 on the bisector of the acute angles the signs of the perpendiculars on the lines are both like or both unlike the signs of the perpendiculars from the origin. If **P** is at P_2 the perpendiculars from P_2 are of opposite sign, one being on the same side of the line as the perpendicular from **O**, the other on the opposite side.

Now consider the equations

$$\frac{3x-4y-6}{5}=\pm\frac{5x+12y-24}{13}$$

They are of the first degree and therefore represent straight lines.

They are satisfied by the co-ordinates (X Y) (see (i.) above), and therefore they are the equations of the bisectors.

One equation is $\frac{3x-4y-6}{5}=\frac{5x+12y-24}{13}$.

i.e. $x-8y+3=0$ bisecting the acute angle.

The other is $\frac{3x-4y-6}{5}=-\frac{5x+12y-24}{13}$.

i.e. $64x+8y-198=0$ bisecting the obtuse angle.

Equation of the Line bisecting the Angle between Two Lines.

Let the equations of the two lines be $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$.

Let P (X, Y) be a point on the line bisecting the angle between the lines, then the perpendiculars drawn from P to both lines must be equal—*i.e.*

$$\frac{a_1\mathrm{X}+b_1\mathrm{Y}+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2\mathrm{X}+b_2\mathrm{Y}+c_2}{\sqrt{a_2^2+b_2^2}}$$

If P is on the bisector of one of the angles—*e.g.* P_1 (Fig. 54)—the signs of the perpendiculars are both like or both unlike the signs of the perpendiculars from O, whereas if P is on the other bisector (*e.g.* at P_2) then only one perpendicular is on the same side as the perpendicular from O.

Now consider the equations:

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

They are of the first degree in x and y and therefore represent straight lines.

They are satisfied by the co-ordinates X, Y, so that the point (X Y) is on them, hence they must be the equations of the two bisectors.

General Equation of the Lines drawn through the Point of Intersection of Two Given Lines.

Let the two given lines be $2x-y=4$ and $3x+2y=5$.

Suppose XY to be their point of intersection. Since this point is on both lines we have

$$2\mathrm{X}-\mathrm{Y}-4=0 \quad \text{(i.)}$$
$$3\mathrm{X}+2\mathrm{Y}-5=0 \quad \text{(ii.)}$$

Now consider the equation $2x-y-4+k\,(3x+2y-5)=0$ where k is a constant (iii.).

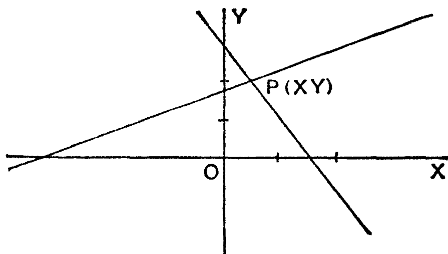


FIG. 55

It is of the first degree and therefore represents a straight line. It is satisfied by the point P (XY) for $2\mathrm{X}-\mathrm{Y}-4+k(3\mathrm{X}+2\mathrm{Y}-5)$ is equal to zero by (i.) and (ii.).

Hence it is the equation of any line through P.

If we require to find the line through P which is perpendicular to the line $4x+5y=20$, we note that the gradient of this line is $-\frac{4}{5}$. But the gradient of (iii.) is $\frac{2+3k}{1-2k}$ and the condition for perpendicularity is $\left(-\frac{4}{5}\right)\left(\frac{2+3k}{1-2k}\right)=-1$.

Hence $k=-\frac{3}{22}$.

$\therefore$ the equation of the required line is $35x-28y-73=0$

EXAMPLES X

1. Find the equations of the lines bisecting the angles between the following pairs of lines:—(i.) $4x=3y$, $3x+4y=0$; (ii.) $y=2x+4$, $2y=x-4$.

2. Find the equations of the internal bisectors of the angles **ABO** and **BAO** where **A** is (4, 0), **B** is (0, 3). Prove that they intersect on the bisector of **BOA**.

3. **P** (4, 4) lies on the parabola $y^2=4x$. **P** is joined to **S** (1, 0) and through **P** a line, **PN**, is drawn parallel to **OX**. Find the equations of the bisectors of the angles between **PN** and **SP**.

4. Find the equations of the bisectors of the angles between the following pairs of lines :—(i.) $4x-3y-1=0$, $3x-4y+2=0$; (ii.) $3x+4y-12=0$, $4x+3y+24=0$.

5. Find the bisectors of the interior angles of the triangle whose sides are $4x+3y-13=0$, $5x+12y+25=0$, $12x-5y-13=0$. Find the co-ordinates of the centre of the inscribed circle of this triangle.

6. Find the equation of the lines passing through the vertices of the triangle whose sides are $2x-3y+1=0$, $x-y=0$, $3x+4y-2=0$ which are (i.) parallel to the opposite sides; (2) perpendicular to the opposite sides. Find the co-ordinates of the orthocentre of this triangle.

7. Find the equations of the internal bisectors of the angles of a triangle whose vertices are (0, 0) $\left(\frac{3}{2}, 2\right)$ $\left(\frac{16}{7}, \frac{20}{21}\right)$.

8. Find the equation of the line through the intersection of $x-2y+3=0$ and $3x+2y-1=0$, which passes through (4, 2).

9. Find the co-ordinates of points on the line $3x-2y=7$ which are equidistant from the lines $3x+4y=0$ and $y+1=0$.

10. Find the equations of the lines parallel to $6x+8y=5$ and distant 2 from it.

11. Find the equation of the line passing through the intersection of $x+y-2=0$ and $x-y+6=0$, and through the intersection of $2x-y+3=0$ and $x-3y+2=0$.

12. Find the locus of points equidistant from $3x-4y+1=0$ and $4x+3y-1=0$.

Loci.

The following examples illustrate further methods of dealing with locus problems:—

1. A point moves so that the difference of the squares

of its distances from two fixed points is constant. Find its locus.

When two fixed points are given the equation of the locus may usually be simplified by taking the line joining the two points as one of the axes, and the perpendicular bisector as the other axis.

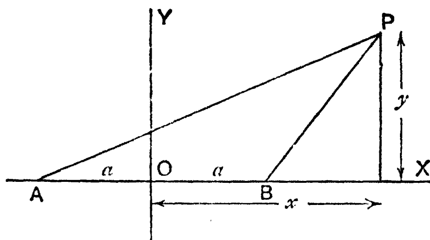


FIG. 56

A and **B** being the two fixed points, take **AB** as **X** axis and **OY** as **Y** axis where **AO**=**OB**=a.

If **P** (x, y) is any point on the required locus, PA^2-PB^2 =constant—*i.e.* $(x+a)^2+y^2-[(x-a)^2+y^2]=k$, or $4ax=k$, *i.e.* $x=\frac{k}{4a}$.

Therefore the locus is a straight line parallel to **OY**.

2. **A** and **B** are two fixed points. Lines **AP**, **BP** are drawn through them such that the gradient of **BP** is twice the gradient of **AP**. Find the locus of **P**.

Take the line joining **AB** as the **X** axis and take the origin midway between **A** and **B**. Take **OA**=**OB**=a, and suppose the co-ordinates of **P** to be (**X**, **Y**) (Fig. 56).

If the gradient of **AP** is m, then the gradient of **BP** is $2m$.

Since **AP** is a line through $(-a, o)$ whose gradient is m, the equation of **AP** is

$$y=m(x+a)$$

Similarly the equation of **BP** is

$$y=2m(x-a)$$

Since **P** (**X**, **Y**) is on both these lines

$$Y=m\,(X+a) \text{ and } Y=2m\,(X-a)$$

The required relation between **X**, **Y** must not contain m

since it is to be true for any value of the gradients; we therefore eliminate m.

Since $$m=\frac{\mathsf{Y}}{\mathsf{X}+a} \text{ and } m=\frac{\mathsf{Y}}{2(\mathsf{X}-a)}$$

$$\therefore \frac{\mathsf{Y}}{\mathsf{X}+a}=\frac{\mathsf{Y}}{2(\mathsf{X}-a)}$$

i.e. $\mathsf{X}=3a$ is the required relation. This shows that **P** lies on a straight line parallel to **OY** at a distance $3a$ from it.

This result can be obtained directly from a figure. The gradient of **AP** is $\frac{\mathsf{Y}}{\mathsf{X}+a}$ and of **BP** it is $\frac{\mathsf{Y}}{\mathsf{X}-a}$.

$$\therefore \frac{\mathsf{Y}}{\mathsf{X}-a}=\frac{2\mathsf{Y}}{\mathsf{X}+a}$$

Note that the equation $\frac{\mathsf{Y}}{\mathsf{X}+a}=\frac{\mathsf{Y}}{2(\mathsf{X}-a)}$ is also satisfied by $\mathsf{Y}=0$. This solution corresponds to the case when $m=0$. Both lines **AP** and **BP** would then lie along **OX** and may be said to meet where $\mathsf{Y}=0$.

EXAMPLES XI

1. **OX**, **OY** are two lines at right angles. A point **P** moves so that the sum of the perpendiculars from it to **OX**, **OY** is constant. What is its locus?

2. A small ring, **R**, runs on a rod **OX**. A string is fastened to a point **O** on the rod, passes through the ring, and its other end **P** is held so that **RP** is always perpendicular to the rod. Find the locus of **P** as the ring moves.

3. A length $\mathbf{AB}=a$ is taken along the **X** axis, and a length $\mathbf{CD}=b$ along the **Y** axis. A point **P** moves so that the sum of the triangles **APB** and **CPD** is constant. Find the equation of the locus of **P**.

4. **OX**, **OY** are two lines at right angles. **A** is on **OX** and **B** on **OY** where $\mathbf{OA}=a$, $\mathbf{OB}=b$. Lines parallel to each other are drawn through **A** and **B** as in Fig. 57. (i.) Prove $\mathbf{OP}.\mathbf{OQ}=ab$. (ii.) Find for what gradient of **PB** and **QA**, **PQ** will be parallel to **AB**.

5. **A** (a, o) and **B** (o, b) are two fixed points on the axes. A point **P** moves along **OB** produced, and **Q** along **OA** produced such that **AQ**$=2$**BP**. Find the locus of the mid-point of **PQ**.

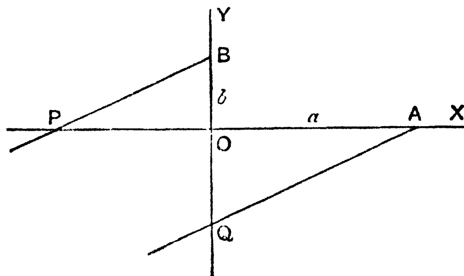


Fig. 57

6. The equation of a certain locus is $x \cos 25° + y \sin 25° = 2$. What do you know about this locus? What is the connection between it and (i.) the locus of the equation $(x-2) \sin 25° - (y-3) \cos 25° = 0$; (ii.) the expression $2 \cos 25° + 3 \sin 25° - 2$.

7. A rectangle **ABCD** (**AB**$=3$ inches, **BC**$=2$ inches) is placed so that **A** moves on **OY** while **B** moves along **OX**. If **OAB**$=\theta$, find the co-ordinates of **D** and find also the locus of **D** by eliminating θ.

8. A string 4 feet long is fastened at one end to a point **B**, and hangs vertically with a weight **P** at the other end. The string passes through a small ring **R** which is held at **B** and then moved horizontally so that **BR** is horizontal and the weight **P** vertically below **R**. Find the locus of **P**.

N.B.—The equation $\frac{x}{a}+\frac{y}{b}=1$ will be found useful in several of the following questions.

9. A straight line moves so that the sum of the reciprocals of its intercepts on two fixed lines which are at right angles to one another is constant and equal to k. Prove that the line passes through the fixed point $\left(\frac{1}{k}, \frac{1}{k}\right)$.

10. A straight line through the fixed point **P** (α, β), cuts the axes at **A** and **B**. If the rectangle **OACB** is completed, find the equation of the locus of **C**.

11. A fixed line **AB** cuts off equal intercepts on the axes. If from **P**, any point on this line, perpendiculars **PM**, **PN** are drawn to **OX**, **OY** respectively, prove that the perimeter of the rectangle **ONPM** is constant. Prove also that the locus of the mid-point of **MN** is a straight line.

12. A square **OABC** is drawn, with **OA** along **OX** and **OC** along **OY**. A string is stretched from a point **P** on **OY** to a point **Q** on **OX**, so that it passes through **B**. Prove that as the string moves the Harmonic Mean of **OP** and **OQ** is constant.

13. A straight line **AB** cuts two lines, **OX**, **OY**, which are at right angles to one another in **A** and **B** respectively. If **OA** +**OB**=10, find the locus of the mid-point of **AB**.

14. **OX**, **OY** are two lines at right angles; **P** is a fixed point (α, β). A straight line with its ends on **OX**, **OY**, always passes through **P** and cuts the axes at **A** and **B**. Find the equation of the locus of the mid-point of **AB**.

15. If a second line through **P** in Qn. 14 cuts the axes at **C** and **D** respectively and **AD**, **BC** are joined, prove that the locus of their point of intersection is a straight line through **O** whose equation is $\beta x + \alpha y = 0$.

16. **A** (a, o), **B** (o, b) are fixed points on the axes. **A** is joined to a point **P** on **OB** and **BM** is drawn perpendicular to **AP**, cutting **AO** in **R**. Find the equation of **PR** and prove that as **P** moves along **OB** the line **PR** moves parallel to itself.

17. Straight lines, **AP**, **BP**, are drawn through two fixed points **A** (a, o), B (b, o) such that the sum of their gradients is unity. Find the locus of **P**.

18. Find the equation of the locus of the centres of circles passing through the point (1, 2) which touch the line $y=o$.

19. On **OX** take two fixed points, **A** and **B**, and on **OY** two other fixed points, **C** and **D**. **P** is a variable point on **BD**. **L** and **M** are the mid-points of **CP** and **AP**. Find the locus of the mid-point of **LM**.

20. **ABC** is a right-angled set square. **AC**=3 inches, **CB**=4 inches, **AB**=5 inches. It is placed with **A** on **OY** and **B** on **OX** so that **C** is on the other side of **AB** from the origin. If **OAB**$=\theta$, find the co-ordinates of **C** and the locus of **C**.

21. **A** and **B** are two points on the **X** axis on opposite sides of **O** such that **OA** = **OB**. **A** and **B** are joined to any point **C**. **CA**, **CB** are bisected at right angles by lines which meet at **I**, find the equation of the locus of **I**.

22. A line through **A** (a, o) meets a line through **A**^{1} $(-a, o)$ at **P**. If the gradient of **AP** equals the reciprocal of the gradient of **A**^{1}**P**, find the locus of **P**.

23. **O** (o, o) and **A** (a, o) are the extremities of the base of an isosceles triangle whose sides have gradients $\pm m$. A point moves so that the sum of the squares of its distances from the three sides of the triangle is constant, find its locus.

24. Through a fixed point **A** (α, β) a straight line is drawn cutting **OX** at **B**. Through **A** draw **AC** at right angles to **AB**, cutting **OY** at **C**. Find the locus of the mid-point of **BC**.

MISCELLANEOUS EXAMPLES

A

1. Plot the three points **P** (1, 1), **Q** (3, 4), **R** (5, −2). Find the area of the triangle **PQR** (i.) by drawing perpendiculars **PM** **QN** from **P** and **Q** to a line through **R** parallel to **OX** and calculating the areas of **PQNM**, **QNR** and **PMR**; (ii.) by finding the lengths of **PR** and the perpendicular from **Q** on **PR**.

2. **P** (−1, 2), **Q** (1, 4), **R** (5, 1) are consecutive corners of a parallelogram. If **S** is the other corner, find the equations of **PS** and **RS**, and the co-ordinates of **S**.

3. Find the tangents to $2x^2+y^2=1$ and $y^2=x$ at their points of intersection, and find the angle at which the curves cut.

4. Find the equation of the locus of a point whose distance from (0, 1) equals its distance from the line $3x-4y=2$.

5. Find the equation of a line which passes through the point **P** (3, 4), such that **OP** is the perpendicular to it from the origin.

B

1. Find the equation of a line drawn perpendicular to $\frac{x}{a}-\frac{y}{b}=1$ through the point where it meets the **X** axis.

2. A straight line makes intercepts of a and b on **OX** and **OY**. Find the co-ordinates of a point **P** on the line whose abscissa equals its ordinate. Hence find the side of a square drawn inside a right-angled triangle when two of its sides lie along the shorter sides, a and b, of the triangle, and one corner lies on the hypotenuse.

3. A point **P** moves so that the perpendiculars from it to $3x+4y=5$ and $12x-5y=13$ are equal. Find the equation of the locus of **P**.

4. Find the co-ordinates of the C. of G. of three equal masses placed at the points (0, 1), (1, 2) and (3, 0).

5. Find the co-ordinates of the points where the circle $x^2+y^2+x-5y-6=0$ cuts the axes, and find the lengths of the chords cut off by the axes.

C

1. Find the equation of a line through (—1, 2) which is parallel to $2x+y=5$.

2. **OA**=12 feet, **OB**=10 feet are two adjacent sides of a rectangular room. Through a point **P**, which is 4 feet from **OB** and 3 feet from **OA**, lines are drawn parallel to the two diagonals of the room. Find where each line meets the other diagonal.

3. A rectangular piece of paper is 12 inches long and 8 inches wide. Find the angle between the diagonals from their equations.

4. Three points, **A**, **B**, **C**, equidistant from each other, are taken on the circumference of a circular lamina which is immersed in water so that the depths of **A**, **B**, **C** are α, β, γ respectively. Find the depth of the centre of the lamina.

5. The three angular points of a triangle are **A** (1, 1), **B** (5, 7), **C** (9, 3). Prove by finding its equation that the line joining the mid-points of **AB** and **AC** is parallel to **BC**.

D

1. The sides **OA**, **OB** of a right-angled triangle are 3 and 4. On both sides of **AB** a square is described. Taking **OA**, **OB** as axes of **X** and **Y**, find in perpendicular form the equations of the sides of the two squares which are parallel to **AB**. Find also the co-ordinates of the angular points of the squares.

2. A point **P** moves so that its distance from (2, 0) is twice its distance from (0, 2). Find the equation of its locus.

3. Find the distance of the point (**X**, **Y**) from the three points **A** (0, 0), **B** (2, 4), **C** (4, 1). Hence find the co-ordinates of the centre of the circle circumscribing **ABC**.

4. A fixed straight line cuts the axis at **A** and **B** where **A** is (a, o), **B** is (o, b). **C** is the mid-point of **AB**, and **CN**, **CM** are drawn perpendicular to the axes. A point **P** moves so that the area **ONPM** equals **AOB**. Find the equation of the locus of **P**.

5. Find the equations of the tangent and normal to $4y=x^3$ at the point whose abscissa is 2. Find the co-ordinates of the point where the tangent cuts the curve.

E

1. A ray of light passes through the point 1, 2, is reflected at a point **A** on the **X** axis and then passes through (5, 3) Find the position of **A** by analysis.

2. A square is circumscribed to the circle $x^2+y^2=9$, so that two of its sides are parallel to $y=2x$. Find the equations of the sides of the square.

3. A length $\mathbf{AB}=a$ is taken on **OX**, and $\mathbf{CD}=b$ on **OY**. A point **P** moves so that $\triangle$ **APB**$-\triangle$**CPD** is constant. Prove that the locus of **P** is a straight line, and find the angle it makes with the line $ax+by+c=0$.

4. Find the equation of the tangent to the circle $x^2+y^2=a^2$ at the point x_1y_1, and show that it can be written in the form $xx_1+yy_1=a^2$. Find the co-ordinates **P**, **Q** of the points of intersection of this tangent with the tangents at (a, o) and $(-a, o)$. Show that **OP** is at right angles to **OQ**.

5. Find the equation of the normal at **P** (x_1y_1) to the parabola $y^2=4ax$, and find the co-ordinates of **G** where it cuts **OX**. If **Q** be the mid-point of **PG** find the locus of **Q**.

1. **A** is a point on **OX** and **B** a point on **OY**. Squares **OAPQ**, **OBLM** are drawn on these lines on the negative side of the axes. If **BP** meets **AL** in **R**, prove that **OR** is perpendicular to **AB**.

2. **A**(a, o), **A**$^1(-a, o)$ are two fixed points. A parallel to **OY** through **A** cuts any line through **A**^{1} at **B**, and a line parallel to **A**^{1}**A** cuts **A**^{1}**B**, **AB** at **R** and **S** respectively. Prove that the mid-point of **RS** lies on **OB**.

3. Find the area of the triangle formed by the lines $y-2x-1=0$, $2y-x-1=0$, $x+y=4$.

4. Find the co-ordinates of the C. of G. and of the orthocentre of the triangle whose angular points are $(-2, 1)$, $(1, 4)$, $(3, -1)$.

5. Find the co-ordinates of points on $2x-y+13=0$, which are at a distance 3 from the line $5x+12y-1=0$.

G

1. **PM** is the perpendicular to **OY** from a point **P** on the curve $y=x^3$ and the normal at **P** meets **OY** in **G**. Prove for various positions of **P** on the curve that $MG \propto \frac{1}{MP}$.

2. **A** and **B** are two fixed points on **OX**. Circles are drawn through **A** and **B** cutting **OY** in **C** and **D**. Find the locus of the point of intersection of **AC** and **BD**.

3. Show that the points on $x^2-4x+y^2-6y+12=0$ for which $\frac{y}{x}$ equals the gradient of the tangent lie on the line $2x+3y-12=0$.

4. A line cuts the axes at **A** and **B** where **OA**$=$**OB**$=4$. On **AB** an equilateral triangle is described whose vertex **C** is on the opposite side of **AB** from the origin. Find the gradients of **AC** and **BC** and their equations.

5. Draw a graph of the curve $xy=2x^2+3$ and determine its asymptotes.

H

1. **AB** meets the axes at **A** $(3, 0)$ **B** $(0, 4)$. On the negative sides of **OA** and **OB** draw squares **OAPQ, OBRS**. Join **AR**, **BP** and find the co-ordinates of **T** their point of intersection. Show that **OT** is perpendicular to **AB** and that it passes through the point of intersection of **PQ** and **RS**.

2. The tangent at **P** to the curve $y^2=4x^3$ meets **OX** in **T** and **OY** in t. If **PN** is the ordinate of **P** prove $3OT=ON$ and $2Ot=PN$.

3. Find the co-ordinates of the point **P** where the perpendicular from the origin meets $9x+12y=25$. If from **Q** $(x_1\ y_1)$ any point on this line perpendiculars **QA, QB** are drawn to the axes find the equations of **AP** and **BP** and show that they are at right angles.

4. Show that if a line drawn from a point $(x_1\ y_1)$ to the point (x_1+a, o) cuts the line $x=a$ at **M** then **OM** will pass through the point $\left(x_1\ \frac{x_1^2 y_1}{a^2}\right)$.

5. Find the co-ordinates of the centre of the circle circumscribing the triangle **A** (4, 6) **B**(0, 4) **C** (6, 2).

CHAPTER V

DETERMINANTS. AREA OF TRIANGLE. MNEMONICS

(This chapter may be omitted on first reading)

When the methods and principles underlying the equations dealing with straight lines have been fully grasped it is of great advantage to be able to write down rapidly the results which have been proved. To do this it is worth while acquiring a slight knowledge of what is known as a determinant.

A determinant of two rows and two columns is written

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

This is understood to mean that a_1 is multiplied by b_2 and b_1 by a_2, the latter product being of negative sign.

It is thus merely another method of writing the expression $a_1b_2-b_1a_2$.

For instance $\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} =(1)\,(4)-(3)\,(-2) =4+6=10$

and $\begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} =(-2)\,(3)-(1)\,(0) =-6$

$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$ is a determinant having 3 rows and 3 columns

It is expanded by multiplying each term in the first row by the determinant obtained by omitting the row and the column in which the term occurs. The signs of the results are alternately positive and negative.

It is therefore equivalent to

$$a_1 \begin{vmatrix} b_2 & 1 \\ b_3 & 1 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & 1 \\ a_3 & 1 \end{vmatrix} + 1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$=a_1(b_2-b_3)-b_1(a_2-a_3)+(a_2b_3-a_3b_2)$$

$$=a_1b_2-a_1b_3-b_1a_2+b_1a_3+a_2b_3-a_3b_2$$

If we expand by taking each term in the first *column* instead of the first *row* we get the same result, thus:

$$a_1 \begin{vmatrix} b_2 & 1 \\ b_3 & 1 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & 1 \\ b_3 & 1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & 1 \\ b_2 & 1 \end{vmatrix}$$

$$=a_1(b_2-b_3)-a_2(b_1-b_3)+a_3(b_1-b_2)$$

$$=a_1b_2-a_1b_3-a_2b_1+a_2b_3+a_3b_1-a_3b_2$$

Solution of Simple Simultaneous Equations.

The solution of the equations $a_1x+b_1y+c_1=0$

$$a_2x+b_2y+c_2=0$$

may be written down at once as follows.

Write the co-efficients only, beginning at the second term, and continue with the co-efficient of x, etc., thus:

$$\begin{matrix} b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{matrix}$$

Now expand these in pairs as three determinants and write the results thus:

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

The solutions of the equations are:

$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} \qquad y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$$

EXAMPLE. Solve $2x-3y=4$, $x+5y=2$.

Writing the equations with the right-hand side zero, we have

$$2x-3y-4=0$$

$$x+5y-2=0$$

Then the co-efficients are

$$\begin{matrix} -3 & -4 & 2 & -3 \\ +5 & -2 & 1 & 5 \end{matrix}$$

and

$$\frac{x}{6+20}=\frac{y}{-4+4}=\frac{1}{10+3}$$

$$\therefore x=\frac{26}{13}=2 \qquad y=0$$

EXAMPLES XII (*a*)

1. Find the value of the determinants:

(i.) $\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix}$ (ii.) $\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix}$ (iii.) $\begin{vmatrix} a & -b \\ b & a \end{vmatrix}$

(iv.) $\begin{vmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 3 & 1 \end{vmatrix}$ (v.) $\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

2. Solve the equations: (i.) $3x+4y=2$, $x-2y=3$; (ii.) $x-4y+1=0$, $2x+3y+2=0$; (iii.) $2x+3y+1=0$, $y-2x=4$; (iv). $y-2x=3$, $3x+y=2$.

Area of a Triangle.

Let the co-ordinates of the angular points be **A** $(-1, 1)$, **B** $(3, 3)$, **C** $(5, -2)$. Through the lowest point **C** draw a parallel to **OX**, and project **AB**, **BC** on it.

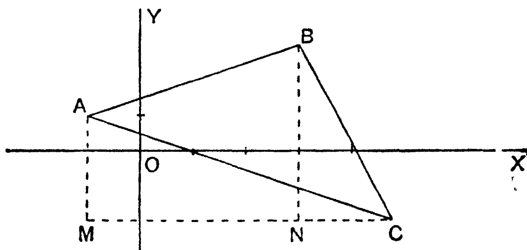


FIG. 58

Then area **ABC** = trapezium **AMNB** + $\triangle$**BNC** − $\triangle$**AMC**.

$$=\frac{1}{2}(3+5)\,4+\frac{1}{2}\,5.2-\frac{1}{2}\,3.6$$

$$=16+5-9=12 \text{ square units}$$

If the co-ordinates are **A** (x_1y_1), **B** (x_2y_2) **C** (x_3y_3) then $\triangle$ **ABC** (Fig. 59).

= trapezium **AMNB** + trapezium **BNRC** − trapezium **AMRC**

$$=\frac{1}{2}(y_1+y_2)(x_2-x_1)+\frac{1}{2}(y_2+y_3)(x_3-x_2)-\frac{1}{2}(y_1+y_3)(x_3-x_1)$$

$$=\frac{1}{2}[y_1x_2-y_2x_1+y_2x_3-x_2y_3+x_1y_3-x_3y_1] \text{ (i.)}$$

If the determinant $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ is expanded it will be found to be equal to (i.) and is therefore a very convenient form for remembering the result. The sign

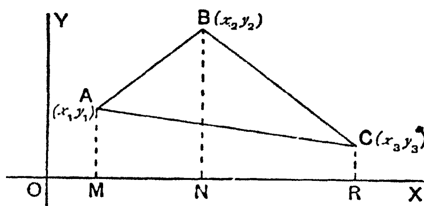


FIG. 59

will depend upon whether we take the points (x_1y_1) (x_2y_2) (x_3y_3) clockwise or anti-clockwise round the triangle.

Equation of the Line joining Two Points.

If the co-ordinates of the two points are (x_1y_1) and (x_2y_2) the required equation is :

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

i.e. $\quad x(y_1-y_2)+y(x_2-x_1)+(x_1y_2-x_2y_1)=0$

This result is easily remembered by the following device :

$$y\uparrow\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\downarrow x$$

Begin with x, the co-efficient of which is y_1-y_2 (shown by the direction of the arrow), then write $y\ (x_2-x_1)$ plus the determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

The equation of the line joining (2, 1), (−3, 4) is

$$y\uparrow\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix}\downarrow x$$

i.e. $\quad x(1-4)+y(-3-2)+(8+3)=0$, or $3x+5y-11=0$.

F

Otherwise. Since P (x, y) and the two given points $(x_1\ y_1)$ $(x_2\ y_2)$ are to lie on a straight line, the area of the triangle formed by these three points will be zero.

i.e.
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

or
$$x(y_1 - y_2) + y(x_2 - x_1) + (x_1 y_2 - x_2 y_1) = 0$$

Equation of the Perpendicular to

$$ax + by + c = 0$$

The gradient of the perpendicular is $\frac{b}{a}$. If the perpendicular is to pass through $(x_1\ y_1)$, its equation is $y - y_1 = \frac{b}{a}(x - x_1)$ which may be written :

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}$$

EXAMPLES XII (*b*)

1. Write down the areas of the triangles whose vertices are : (i.) (3, 2), (5, 6), (1, 4) ; (ii.) (2, 1), (3, −2), (4, −1) ; (iii.) (0, 0), (a, b), $(-b, a)$; (iv.) (7, 6), (0, 0), (4, 3) ; (v.) (−1, 3), (−1, −1), (2, 1) ; (vi.) (−3, 1), (−2, −2), (3, 2).

2. Prove that the following points lie on a straight line by showing that the area of the triangle ABC is zero :—(i.) A (2, 3), B (4, 5), C (6, 7) ; (ii.) A (3, 0), B (1, 8), C (4, −4).

3. Write down the equations of the lines joining the following pairs of points :—(i.) (2, 3), (−4, 1) ; (ii.) (2, 1), (6, 6) ; (iii.) (−5, 2), (7, 0) ; (iv.) (1, −2), (2, 2) ; (v.) $(-a, o)$, (a, b) ; (vi.) (−2, 3), (2, 1).

CHAPTER VI

CONIC SECTIONS

Historical Note.

If a right-angled triangle, **ABC**, is rotated about **AB** the surface generated by **AC** will be a right circular cone. If **CA** is produced beyond **A** another cone, $\mathbf{C^1AD^1}$, will be formed, and the solid is called a double cone. The plane curves formed by the intersection of this solid with a plane will vary in shape according to the position of the intersecting plane. A section which cuts both generating lines, **AC** and **AD**, forms a closed curve, **EF**, called an ellipse: if the section is perpendicular to the axis **AB** the curve becomes a circle, **DC**. When the plane is parallel to **AD** the curve is confined to the lower half cone but is not closed, since it spreads out continuously as the generating lines are produced: the curve is then called a parabola (**KGL**).

If the plane cuts both parts of the cone, there are two detached branches, **SPM**, $\mathbf{S^1P^1M^1}$, forming a curve called a hyperbola.

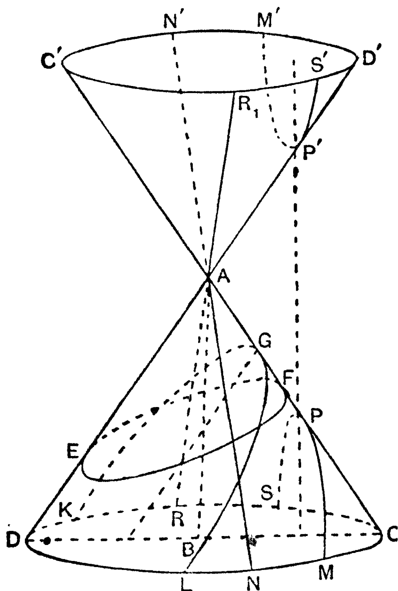


FIG. 60

If the plane passes through the vertex **A** the section consists of two straight lines, $\mathbf{NAN^1}$, $\mathbf{RAR^1}$.

The names of the conic sections, ellipse, parabola and hyperbola, are due to Apollonius of Perga, third century B.C.

Since the investigation of the properties of these curves involved

the drawing of a solid figure it was difficult to find any general relations connecting them, and it was not until about A.D. 300 that Pappus of Alexandria discovered that the curves might be described by a point which moved so that its distance from a fixed point bore a constant ratio to its distance from a fixed line.

For nearly two thousand years the properties of these curves were studied merely as an abstract science, without any idea of their practical importance. In 1571 Kepler, who was an astronomer as well as a geometer, enunciated the theory that the paths of the planets round the sun were ellipses, and the verification of this law was an essential step in the proof of Newton's theory of universal gravitation.

A fitting climax to these developments was provided by Descartes, who, after showing that every equation of the first degree of the form $ax+by+c=0$ represented a straight line, proceeded to find the geometrical representation of the general equation of the second degree—namely $ax^2+2hxy+by^2+2gx+2fy+c=0$. We can imagine how great must have been his delight and surprise to find that it represented no more and no less than the conic sections which for generations had been studied as a branch of geometry. When $h=0$ and $a=b$ the equation represents a circle, if $ab-h^2$ is positive it represents an ellipse: when $ab-h^2=0$ it is a parabola: when $ab-h^2$ is negative the curve is a hyperbola, and when the expression can be split up into two linear factors it represents two straight lines.

A Pair of Straight Lines.

When an equation of the second degree in x and y can be factorised, it splits up into two simple equations, each of which represents a straight line.

The equation $x^2-y^2=0$ can be written in the form $(x-y)\ (x+y)=0$ and therefore represents the two lines $x-y=0$ and $x+y=0$.

$3x^2+7xy+2y^2-2x+y-1=0$ may be factorised thus: $3x^2+7xy+2y^2=(3x+y)\ (x+2y)$, and since the factors of the given expression will have three terms of which the third is numerical, inspection shows that this term will be $+1$ in one factor and -1 in the other. Trial now shows that the required factors are $3x+y+1$ and $x+2y-1$, so that the equation represents the two lines $3x+y+1=0$ and

$x+2y-1=0$. Any point—*e.g.* $(0, -1)$, which satisfies $3x+y+1=0$—will necessarily satisfy

$$3x^2+7xy+2y^2-2x+y-1=0 \quad \text{(i.)}$$

Similarly any point whose co-ordinates satisfy $x+2y-1=0$ will also satisfy (i.). $\therefore$ (i.) represents both the lines $3x+y+1=0$ and $x+2y-1=0$.

The Circle.

When the centre is at the origin the equation of a circle has been shown to be $x^2+y^2=r^2$ (p. 9).

Change of Axes.

If the centre of the circle is at **C** whose co-ordinates are (α, β), we can find its equation by moving the axes from **C** to **O**, keeping them parallel to themselves.

Suppose the co-ordinates of **P**, any point on the circle, to be (x, y) referred to axes through **C**, and to be $(\mathbf{X}, \mathbf{Y})$ referred to parallel axes through **O**.

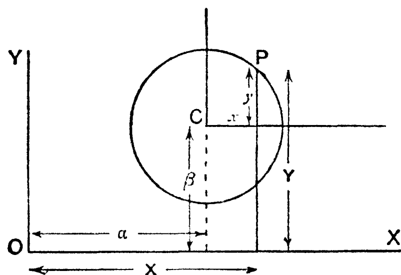


Fig. 61

From the figure $x=\mathbf{X}-\alpha$ and $y=\mathbf{Y}-\beta$. But we know that $x^2+y^2=r^2$.

$\therefore$ $(\mathbf{X}-\alpha)^2+(\mathbf{Y}-\beta)^2=r^2$, which is the equation of the circle referred to axes through **O**.

If we write x and y for the co-ordinates of a point referred to axes through **O** the equation is

$$(x-\alpha)^2+(y-\beta)^2=r^2$$

Expanding we have

$$x^2+y^2-2x\alpha-2y\beta+\alpha^2+\beta^2-r^2=0 \quad \text{(i.)}$$

It will be noticed in this equation (i.) that the co-efficients of x^2 and y^2 are equal; (ii.) that there is no term in xy.

These facts are peculiar to the equation of a circle and enable us to identify the curve at once from its equation.

In three dimensions the general homogeneous equation of the second degree in x, y, z is written:

$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy=0$$

If we put $z=1$ in this it reduces to

$$ax^2+by^2+c+2fy+2gx+2hxy=0$$

This is called the general equation of the second degree in x and y.

For a circle $a=b$, and for simplicity a and b are usually made equal to 1; also $h=0$.

The general equation of a circle is therefore

$$x^2+y^2+2gx+2fy+c=0$$

It will be seen that this represents all the terms that occur in (i.), c standing for the numerical part $\alpha^2+\beta^2-r^2$.

EXAMPLE 1. Find the co-ordinates of the centre of the circle $x^2+y^2+2gx+2fy+c=0$.

This has to be written in the form $(x-\alpha)^2+(y-\beta)^2=r^2$ —that is, the terms in x and the terms in y have to be made into perfect squares, thus:

$$x^2+2gx+g^2+y^2+2fy+f^2=g^2+f^2-c$$

i.e. $$(x+g)^2+(y+f)^2=g^2+f^2-c$$

$\therefore$ $\alpha=-g$; $\beta=-f$ so that the co-ordinates of the centre are $(-g, -f)$, and the radius is $\sqrt{g^2+f^2-c}$.

EXAMPLE 2. Find the centre of the circle:

$$2x^2+2y^2+5x-2y=0$$

To get this in the required form we must make the coefficients of x^2 and y^2 equal to 1, thus:

$$x^2+y^2+\frac{5x}{2}-y=0$$

Completing the squares, we have

$$x^2+\frac{5}{2}x+\frac{25}{16}+y^2-y+\frac{1}{4}=\frac{25}{16}+\frac{1}{4}$$

$$\therefore \left(x+\frac{5}{4}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{29}{16}$$

$\therefore$ $\alpha=-\frac{5}{4}$, $\beta=\frac{1}{2}$ are the co-ordinates of the centre.

Example 3. Find the equation of the circle passing through the three points, **A** (0, 1), **B** (3, 0), **C** (0, 6).

The equation of any circle is of the form

$$x^2+y^2+2gx+2fy+c=0$$

Since **A** is on it we have

$$0+1+0+2f+c=0 \qquad \text{Similarly for } \mathbf{B} \text{ and } \mathbf{C}.$$
$$9+0+6g+0+c=0$$
$$0+36+0+12f+c=0$$

Solving, we find $g=-\frac{5}{2}$, $f=-\frac{7}{2}$, $c=6$.

Hence the equation is $x^2+y^2-5x-7y+6=0$.

Note that there are 3 unknowns and that 3 points are necessary to fix the circle.

Length of a Tangent to a Circle.

If the co-ordinates of the centre are (α, β) and the radius r, the equation of the circle is

$$(x-\alpha)^2+(y-\beta)^2=r^2$$

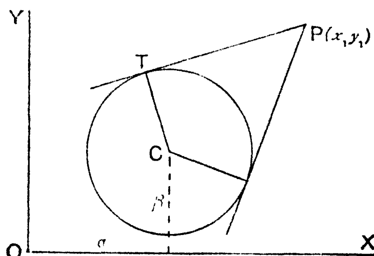


Fig. 62

If **PT** is the tangent from (x_1y_1) to the circle we have

$$\mathbf{PC}^2=(x_1-\alpha)^2+(y_1-\beta)^2$$

but $\mathbf{PT}^2=\mathbf{PC}^2-r^2$

$$=(x_1-\alpha)^2+(y_1-\beta)^2-r^2$$

The square of the tangent is therefore obtained by substituting the co-ordinates of the point x_1y_1 in the expression $(x-\alpha)^2+(y-\beta)^2-r^2$.

Show by writing the equation $x^2+y^2+2gx+2fy+c=0$ in

the form $(x-\alpha)^2+(y-\beta)^2=r^2$ that the square of the tangent from (x_1y_1) is

$$x_1^2+y_1^2+2gx_1+2fy_1+c$$

Note that the result will be zero if (x_1y_1) is on the curve.

EXAMPLES XIII

1. State what loci are represented by the following equations :—(i.) $xy=0$; (ii.) $(x-2)(x-1)=0$; (iii.) $x^2-5xy+4y^2=0$.

2. Show that the following equations represent pairs of straight lines, and find the angles between them :—(i.) $x^2-5xy+4y^2+3x-4=0$; (ii.) $3x^2+5xy+2y^2-4x-3y+1=0$; (iii.) $xy-ax-by+ab=0$.

3. Find the equation of the circle whose centre is **P** and radius r when (i.) **P** is (2, 1), $r=2$; (ii.) **P** (—3, 2), $r=4$.

4. Find the co-ordinates of the centre and the radius of the following circles :—(i.) $(x+2)^2+y^2=9$; (ii.) $x^2+y^2-6x=4$; (iii.) $x^2+y^2=0$; (iv.) $x^2-x+y^2+y=0$; (v.) $16x^2-16x+16y^2+8y=11$; (vi.) $7x^2+7y^2-4x-y=3$.

5. Find the equations of the following circles :—(i.) centre (2, 3) passing through (3, —2) ; (ii.) through the points (4, 2), (—6, —2) with centre on **OY** ; (iii.) centre (—1, —5) and touching **OX** ; (iv.) through (0, —3) (4, 0) with centre on the line $x+2y=0$.

6. Find the equations of the circle $x^2+y^2=9$, if the centre is transferred to the point (i) (2, 3) ; (ii) (—2, 3).

7. Find the equations of the following circles when the origin is moved to their centres :—(i.) $(x-2)^2+(y+4)^2=9$; (ii.) $x^2-4x+y^2+8y=12$.

8. **ABC** is a triangle right-angled at **B**. **BA**$=4$, **BC**$=3$. Find (i.) the equation of the side **AC** ; (ii.) the length of the perpendicular from a point (α, β) to this side ; (iii.) the equation and radius of the inscribed circle. (Take **B** as origin and **BC** as **X** axis.)

9. Find the locus of a point which moves so that the sum of the squares of its distances from (a, o) and $(-a, o)$ is equal to $2b^2$. What happens when $b=a$?

10. Find the locus of a point such that the square of its distance from the origin is equal to its distance from the X axis multiplied by a constant k. Find the centre of the curve.

11. Find the equation of a circle whose centre is at the point (4, 5) and whose circumference passes through the centre of the circle $x^2+y^2+4x-6y=12$.

12. Form a single equation to represent the two straight lines, $3y-2x=1$, $5y+4x=2$. Find also the equation of the two straight lines joining the origin to the points where these lines are met by the line $y=\frac{2}{11}$.

13. Two circles are drawn to touch both the lines $y=0$, $y=x \tan 2\alpha$, and to pass through the point (h, k). Prove that the distances of the points of contact with the X axis from the origin are the roots of the equation $X^2-2(h+k \tan \alpha) X+h^2+k^2=0$.

14. Show that the four straight lines given by the equations $6x^2-5xy-6y^2=0$ and $6x^2-5xy-6y^2+x+5y-1=0$ lie along the sides of a square.

15. A circle touches one given straight line and cuts off a constant length $2l$ from another straight line perpendicular to the former. Find the equation of the locus of its centre.

16. Write down the equations to the straight lines joining XY to the points x_1y_1 and x_2y_2. Then write down the condition that these lines are at right angles, and hence obtain the equation of the circle described on the line (x_1y_1) (x_2y_2) as diameter.

17. Find the equations of the two lines given by $2x^2-3xy-2y^2+2x+11y-12=0$ and find also in quadratic form the equation of the two lines parallel to them which pass through the origin.

18. Find the equation of the line joining the centres of the circles $x^2-4x+y^2+2y+1=0$ and $16x^2-16x+16y^2+8y=11$.

19. Show by analysis that the four points (16, 1) (16, 3) (14, 7), (2, 3) are concyclic.

20. Find the co-ordinates of the points on the circle $x^2-6x+y^2-2y+6=0$ which are equidistant from the axes.

21. Find the square of the distance between the centres of the circles $x^2+y^2+4x+6y+4=0$, $x^2+y^2-6x-2y-22=0$, and prove that it equals the sum of the squares of the radii. Hence show that the circles cut at right angles—*i.e.* orthogonally.

22. Show that the circles $x^2+y^2-2ax+c=0$ and $x^2+y^2+2by-c=0$ intersect orthogonally.

23. Show that if the distance between the centres of two circles is a, the obtuse angle θ at which they intersect is given by $2r_1r_2 \cos \theta=r_1{}^2+r_2{}^2-a^2$. Hence find the angle at which the circles $x^2-8x+y^2-2y+13=0$, and $x^2+2x+y^2+2y-34=0$ intersect.

24. Find the length of the tangent from (4, 7) to

$$x^2-4x+y^2-6y+12=0$$

If a point moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed line, the curve described is either a parabola, ellipse or hyperbola. The constant ratio is known as the eccentricity, and denoted by e.

If $e=1$ the curve is a Parabola.

If $e<1$ the curve is an Ellipse.

If $e>1$ the curve is a Hyperbola.

The fixed point is called the Focus of the curve and the fixed line is called the Directrix.

The Parabola.

On squared paper take a point S, 2 units from a straight line OY.

If SO is perpendicular to OY, the mid-point A of SO is such that $\frac{\text{SA}}{\text{AO}}=1$.

$\therefore$ A is a point on the parabola.

Mark the parallels to OY at unit distance apart, in order, 1, 2, 3, etc.

With centre **S**, radius 2 units, mark the points P_1, P_2,

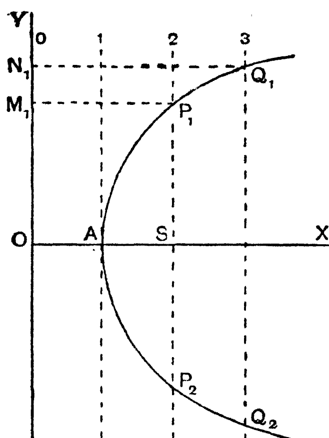


FIG. 63

where an arc cuts the line 2, and with radius 3 mark the points Q_1, Q_2 where it cuts the line 3, etc.

Then $\qquad \dfrac{SP_1}{P_1M_1}=1, \quad \dfrac{SQ_1}{Q_1N_1}=1$

A curve through the points A, P_1 P_2 Q_1 Q_2, etc., will be a parabola.

The foot of the perpendicular from **S** to the directrix is usually called **Z**.

The curve is symmetrical about the line **AX**, which is called the axis of the parabola.

The distance **SZ** is called $2a$, so that $SA=AZ=a$.

When **Z** is taken as origin, **ZS** as **X** axis and the directrix as **Y** axis, then if

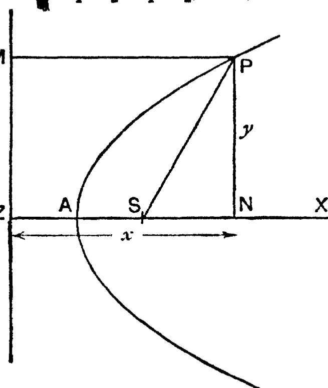


FIG. 64

P (x, y) be any point on the curve it can be shown as on p. 10, that since SP=PM we get $y^2=4a(x-a)$ for the equation of the parabola.

Since $x-a=$ AN and AS$=a$ this relation is PN$^2=4$ AN, AS.

A simpler form of the equation is obtained by taking A as the origin, and the tangent to the curve at A as Y axis.

FIG. 65

Then AN$=x$, NZ$=$PM$=x+a$, SN$=x-a$.

Now $\frac{SP}{PM}=1$ $\therefore SP^2=PM^2=NZ^2$

$$\therefore y^2+(x-a)^2=(x+a)^2$$

$$\therefore y^2=4ax$$

The double ordinate through the focus (LSL^1) is called the Latus Rectum. Since SL $=$ LR $=$ SZ $=2a$, the Latus Rectum$=4a$.

FIG. 66

If the axis of the parabola is taken as the Y axis and the tangent at A as the X axis, show that the equation of the curve is now $x^2=4ay$ (Fig. 66).

If $a=\frac{1}{4}$, this reduces to the form $y=x^2$.

It is important to notice that in the equation $y^2=4ax$ any positive value of x determines *two* equal and opposite values for y, and that x cannot be negative. Similarly for $x^2=4ay$ *two* values of x correspond to each positive value of y.

Change of Axes.

If the vertex **A** is at the point $(\alpha\beta)$, the axes of co-ordinates being parallel to their original directions, we see that if $(\mathbf{X}\mathbf{Y})$ are the co-ordinates of any point **P** referred to axes through **O**, then $x=\mathbf{X}-\alpha$, $y=\mathbf{Y}-\beta$ (*cf.* p. 77).

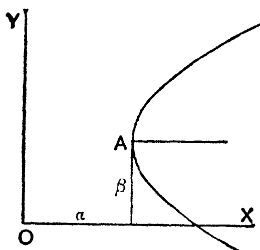


FIG. 67

But since $y^2=4ax$ we have $(\mathbf{Y}-\beta)^2=4a\ (\mathbf{X}-\alpha)$.

If we take the co-ordinates of **P** referred to axes through **O** as x and y the equation of the parabola is $(y-\beta)^2=4a\ (x-\alpha)$.

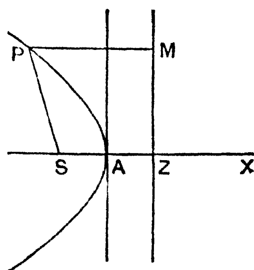


FIG. 68

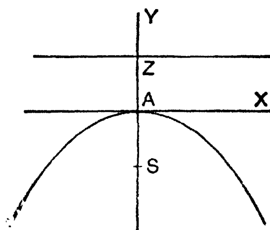


FIG. 69

Prove in a similar way that if the axis of the parabola is parallel to the Y axis the equation of the curve is $(x-\alpha)^2=4a\,(y-\beta)$.

If the parabola is concave towards the left the relation between x and y will be the same as before except that x is now negative and the equation will be $y^2=-4ax$ when the vertex is at the origin (Fig. 68).

Similarly if the parabola is concave downwards its equation will be $x^2=-4ay$ (Fig. 69).

Axis of Parabola parallel to OX.

We have proved that the equation is

$$(y-\beta)^2=4a(x-\alpha)$$

i.e.

$$y^2-2y\beta+\beta^2=4ax-4a\alpha$$

or

$$x=\frac{y^2}{4a}-\frac{2\beta}{4a}y+\left(\frac{\beta^2+4a\alpha}{4a}\right)$$

This may be written in the general form: $x=ay^2+by+c$, where a, b, c are constants.

Note that a in this equation is not the same as SA$=a$ in the original form.

Axis of Parabola parallel to OY.

The equation is $(x-\alpha)^2=4a\,(y-\beta)$.

On expansion this reduces to the form $y=ax^2+bx+c$.

Example. To find the co-ordinates of the vertex of the parabola $2y^2+3y+4x=2$.

This must be written in the form $(y-\beta)^2=4a\,(x-\alpha)$.

Making the co-efficient of y^2 unity we have

$$y^2+\frac{3}{2}y=-2x+1$$

$$\therefore\ y^2+\frac{3}{2}y+\frac{9}{16}=-2x+\frac{25}{16}$$

i.e.

$$\left(y+\frac{3}{4}\right)^2=-2\left(x-\frac{25}{32}\right)$$

$$\therefore\ \alpha=\frac{25}{32}\quad \beta=-\frac{3}{4}$$

If the origin were at the vertex—*i.e.* at the point $\left(\frac{25}{32}, -\frac{3}{4}\right)$, the equation would become $y^2=-2x$.

The parabola is therefore concave to the left, and since $a=\frac{1}{2}$ the focus is at the point $\left(\frac{9}{32}, -\frac{3}{4}\right)$.

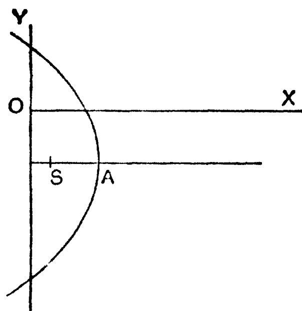


FIG. 70

The co-ordinates of the vertex may easily be found by finding $\frac{dx}{dy}$ from the equation of the curve.

Since $\quad 2y^2+3y+4x=2 \qquad 4y+3+4\frac{dx}{dy}=0$

$$\therefore \frac{dx}{dy}=-\frac{4y+3}{4}$$

At the vertex, since the gradient of the tangent is ∞

$$\therefore \frac{dy}{dx}=\infty. \qquad \therefore \frac{dx}{dy}=0$$

$$\therefore \text{ at the vertex } 4y+3=0 \text{ or } y=-\frac{3}{4}$$

By substituting in the equation of the curve we find $x=\frac{25}{32}$.

To find the co-ordinates of the vertex of $y=ax^2+bx+c$.

This has to be written in the form

$$(x-\alpha) = 4a\,(y-\beta)$$

We have $$x^2+\frac{b}{a}x=\left(\frac{y}{a}-\frac{c}{a}\right)$$

$$\therefore\ x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{y}{a}+\frac{b^2-4ac}{4a^2}$$

$$\therefore\ \left(x+\frac{b}{2a}\right)^2=\frac{1}{a}\left[y+\frac{b^2-4ac}{4a}\right]$$

$$\therefore \text{ the vertex is } -\frac{b}{2a},\ -\frac{b^2-4ac}{4a}$$

Otherwise at the vertex the gradient of the tangent is zero.

$$\therefore\ \frac{dy}{dx}=2ax+b=0$$

$$\therefore\ x=-\frac{b}{2a}$$

By substitution we find $y=-\dfrac{b^2-4ac}{4a}$.

General Equation of the Parabola.

Suppose the focus to be at the point (2, 3) and the directrix to be the line $3x+4y=12$.

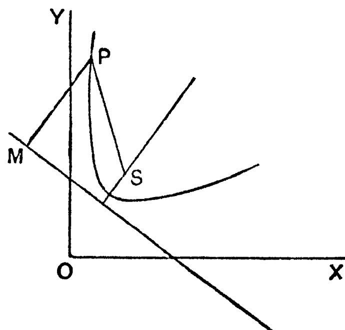


FIG. 71

If P (X Y) be any point on the curve

$$SP^2=(X-2)^2+(Y-3)^2$$

Also the perpendicular from P to the directrix is $\frac{3X+4Y-12}{5}$ and since $SP^2=PM^2$

$$\therefore (X-2)^2+(Y-3)^2=\left(\frac{3X+4Y-12}{5}\right)^2$$

or $$16X^2+9Y^2-24XY-28X-54Y+181=0$$

$\therefore$ XY lies on the curve

$$16x^2+9y^2-24xy-28x-54y+181=0$$

It will be noticed that the terms of the second degree $16x^2+9y^2-24xy$ form a perfect square, and this is characteristic of all equations of the parabola.

The general equation of a parabola is

$$(ax+by)^2+2gx+2fy+c=0$$

EXAMPLE. Find the parabola with axis parallel to OY which passes through the points (0, 0), (10, 12), (30, 8).

The form of the equation will be $y=ax^2+bx+c$.

Since it passes through the three given points we have $0=c$, $12=100a+10b+c$, $8=900a+30b+c$.

Hence $a=-\frac{7}{150}$ $b=\frac{5}{3}$, and the equation is

$$y=-\frac{7}{150}x^2+\frac{5}{3}x$$

EXAMPLES XIV

1. Find the equations of the following parabolas :—(i.) Focus at 2, 0. Directrix $x=0$; (ii.) Focus at 4, 0. Directrix $x=-4$; (iii.) Focus at 0, 4. Directrix $y=0$. (iv.) Focus at 1, 2. Directrix $3x-4y=5$. (v.) Focus $(-2, 0)$. Directrix $x=2$. (vi.) Focus $(0, -2)$. Directrix $y=2$; (vii.) Focus 0, 0. Directrix $y=4$.

2. Find the equations of the following parabolas :—(i.) Focus (2, 2). Directrix $x=-2$; (ii.) Focus $(-2, 2)$. Directrix $x=0$; (iii.) Focus (1, 2). Directrix $y=4$; (iv) Focus (1, 2). Directrix $y=-4$.

3. Find the co-ordinates of the vertex and focus and the equation of the directrix for the following parabolas :—(i.) $y=x^2+x$; (ii.) $y=3x^2-2x$; (iii.) $x=2y^2-4y+1$; (iv.) $y^2+2x-y+2=0$; (v.) $x^2+6x-y+4=0$.

4. Find the equation of the tangent to $y=x^2$ at the point **P** on the curve whose abscissa is 2. If this tangent cuts the **Y** axis at **T**, find the co-ordinates of **T**. Find the length **SP** and prove that it equals **ST**.

5. Find the equation of the normal to $y^2=4x$ at **P** (1, 2). If it meets the **X** axis at **G** find the co-ordinates of **G**. If **PN** is the ordinate of **P**, find the length of **NG**.

6. Plot the system of curves given by $y^2=2kx$ when $k=\frac{1}{2}$, 1, 2, ∞, -1, 0.

7. Write down the equations of the following parabolas if the vertex were at the origin, the axes being parallel to their original directions :—$y^2+2y+x+4=0$, $4x^2+x-y+2=0$.

8. The cable of a suspension bridge is a parabola. The roadway is 5 feet below the lowest point of the cable. The span of the bridge is 50 feet, and the tops of the piers are 15 feet above the roadway. Find the S.L.R. of the parabola. If the road is supported by vertical chains at intervals of 5 feet, find their lengths.

9. The girder of a railway bridge is a parabola, with its vertex at the highest point, 10 feet above the ends. If the span is 100 feet, find its height at 10 feet, and 20 from the mid-point.

10. The curve described by a projectile when air resistance is neglected is a parabola whose equation is of the form $y=a+bx+cx^2$. Find the constants in this equation if a ball is projected from the origin at an angle of 45° and hits the ground 20 feet away. Find the highest point reached and the height of the ball when it has travelled 5 feet horizontally.

11. Find the angle of projection of a body whose path is a parabola and which, starting from the point (0, 4), passes through (20, 6), and (80, 0). How high is the body when it has travelled 40 feet horizontally ?

12. Prove that a graph showing the relation between the area of a circle and its radius is a parabola.

13. If the air resistance **R** to a projectile partly varies as the velocity of flight (v) and partly as v^2 show that the graph giving the relation between **R** and v is a parabola. Find the co-ordinates of its vertex if the constants are k and λ respectively.

14. Plot the system of curves given by $y^2=2kx+k^2$ as k varies (positive or negative).

15. Find the equations of the parabolas given (i.) Focus (1, 1). Directrix $3x-4y=2$; (ii.) Focus (2, -1). Directrix $4x+3y=5$. Find the equation of the axis of each of the above parabolas.

16. Find the co-ordinates of the point of intersection of the axis and the directrix of the parabola whose focus is at (2, 2) and directrix the line $3x-4y=5$. Find also the length of its Latus Rectum.

17. Find the equation of the tangent to $y^2=4ax$ at x_1y_1, and prove that it can be written in the form $yy_1=2a(x+x_1)$.

The Ellipse.

On squared paper take a point S, 3 units from a line XD, and let the eccentricity of the ellipse be $\frac{1}{2}$.

If $XA=2$ then $SA=1$. $\therefore \frac{SA}{AX}=\frac{1}{2}$ and A is a point on the curve. Also if $SA^1=3$ then $A^1X=6$. $\therefore \frac{SA^1}{A^1X}=\frac{1}{2}$ and A^1 is another point on the curve.

Draw an arc with S as centre and radius 1·5, cutting the line 3 at P_1, P_2, also an arc with radius 2 cutting 4 at

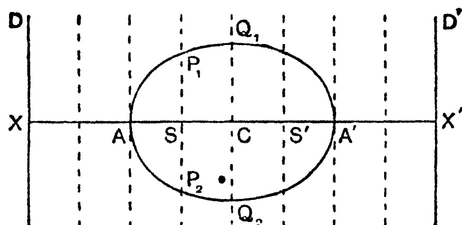


Fig. 72

Q_1, Q_2 . . . and so obtain a series of points on the curve.

Bisect AA^1 at C and mark off $S^1A^1=SA$, $A^1X^1=AX$.

Then we can describe the same curve by taking S^1 as the focus and X^1D^1 as the directrix.

To prove $CS = e\ CA$
$CA = e\ CX$

Since $SA = e\ AX$ (i.)

and $SA^1 = e\ A^1X$ (ii.)

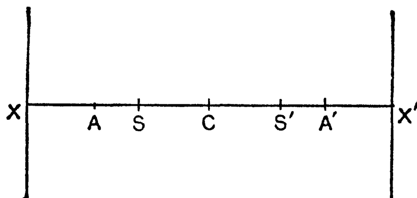


FIG. 73

By addition $SA + SA^1 = e\ (AX + A^1X)$

i.e. $AA^1 = e\ XX^1$ since $AX = A^1X^1$

$$\therefore\ CA = e\ CX$$

CA is usually called a $\qquad \therefore\ CX = \frac{a}{e}$

By subtracting (i.) and (ii.) $SS^1 = e\ AA^1$ since $S^1A^1 = SA$

$$\therefore\ CS = e\ CA = ea$$

Equation of the Ellipse.

Through C the centre of the curve draw YCY^1 at right angles to XX^1.

Take C as origin and CX^1, CY as axes.

Let $P\ (x, y)$ be any point on the curve, then;

$$SP^2 = (ae + x)^2 + y^2$$
$$PM^2 = (XC + CN)^2$$
$$= \left(\frac{a}{e} + x\right)^2$$

$\therefore$ since $SP = e\ PM$ we have

$$(ae + x)^2 + y^2 = e^2\left(\frac{a}{e} + x\right)^2$$

whence $\qquad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

or $\qquad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$

If the Y axis cuts the curve at B the length CB is called b, and since the point (o, b) is on the curve

$$\frac{o}{a^2}+\frac{b^2}{a^2(1-e^2)}=1 \qquad \therefore\ b^2=a^2(1-e^2)$$

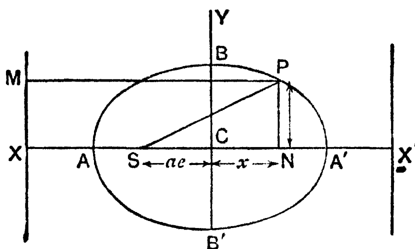


FIG. 74

The equation of the curve then becomes $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

The relation $b^2=a^2(1-e^2)$ enables us to find e when a and b are known. AA^1 is called the major axis of the curve and BB^1 the minor axis.

Semi-Latus Rectum.

If SL is the ordinate of the point L which is on the curve

$$SL=e\ LM=e\ SX=e\ (CX-CS)$$
$$=e\left(\frac{a}{e}-ae\right)$$
$$=a-ae^2=a\ (1-e^2)$$
$$=\frac{b^2}{a}$$

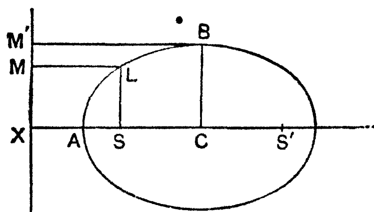


FIG. 75

Construction for Foci.

Since B is on the curve $SB = e\ BM^1 = e\ (CX) = e.\frac{a}{e} = a.$

Therefore a circle with centre B and radius a cuts the major axis at the foci.

Mechanical Construction of the Curve.

If P is any point on the curve

$$SP = e\ PM,\quad S^1P = e\ PM^1$$

$$\therefore SP + S^1P = e\ (PM + PM^1) = e\ XX^1$$

$$= 2e\ CX = 2e\ \frac{a}{e} = 2a$$

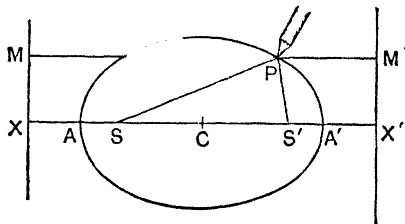


FIG. 76

$$\therefore SP + S^1P \text{ is constant} = AA^1$$

If then a thread of length $2a$ is fastened at S and S^1 and kept tightly stretched by a pencil at P, the pencil as it moves will trace out an ellipse.

Change of Axes.

If the centre is at $(\alpha\beta)$ the axes of co-ordinates being parallel to the axes of the ellipse, we have, as before (p. 77), $x = X - \alpha$; $y = Y - \beta$.

$$\therefore \frac{(X-\alpha)^2}{a^2} + \frac{(Y-\beta)^2}{b^2} = 1$$

Thus the equation of an ellipse with centre at. $(\alpha\beta)$ and axes parallel to the axes of co-ordinates is

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$$

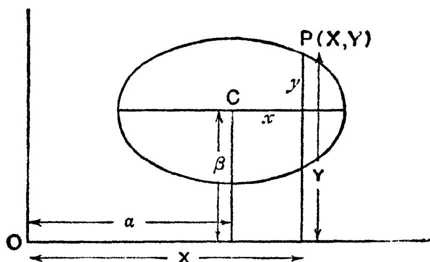


FIG. 77

If the major axis of the ellipse lies along the Y axis of co-ordinates the equation of the curve is obtained by interchanging x and y, and is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

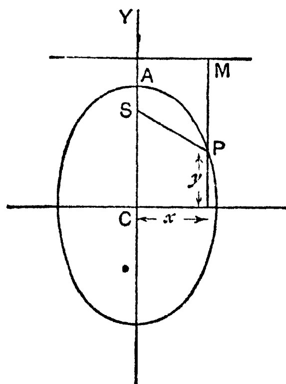


FIG. 78

Note that when $y=0$, $x=\pm b$, and when $x=0$, $y=\pm a$.

Auxiliary Circle.

If a circle is described on AA^1 as diameter its equation will be $x^2+y^2=a^2$; this circle is called the auxiliary circle.

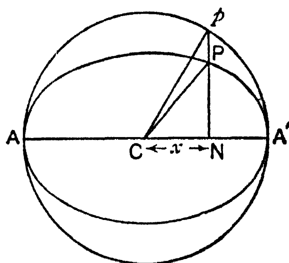


FIG. 79

If p is a point on this circle where $CN=x$ we have $pN=\sqrt{a^2-x^2}$.

If P is the corresponding point on the ellipse

$$\frac{x^2}{a^2}+\frac{PN^2}{b^2}=1 \qquad \therefore\ PN=\frac{b}{a}\sqrt{a^2-x^2}$$

Hence
$$PN=\frac{b}{a}pN$$

Eccentric Angle.

If the angle pCN is θ we have $x=a\cos\theta$ and $pN=a\sin\theta$

$$\therefore\ PN=\frac{b}{a}pN=b\sin\theta$$

Hence the co-ordinates of P on the ellipse may be written $a\cos\theta$, $b\sin\theta$. θ is called the eccentric angle of the point P.

Projection of a Circle into an Ellipse.

On a plane which is inclined to the horizontal at an angle θ given by $\cos\theta=\frac{b}{a}$, describe a circle, and project this circle

on to the horizontal plane. A diameter AA^1 drawn parallel to BC will project into aa^1 where $AA^1 = aa^1$. Any line pn

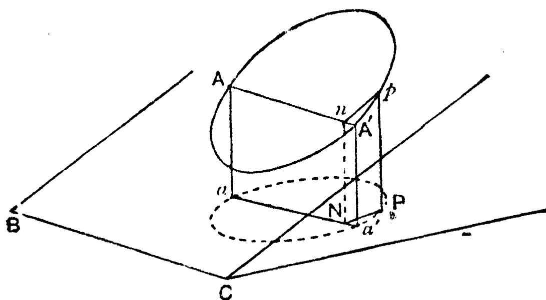


FIG. 80

perpendicular to AA^1 will project into PN where $PN = pn \cos\theta = pn\frac{b}{a}$.

The projection of the circle is therefore an ellipse whose auxiliary circle is equal to the circle on the inclined plane.

Area of an Ellipse.

If a series of rectangles such as PM and pM is drawn in the ellipse and its auxiliary circle respectively, the area $PM = \frac{b}{a}pM$ since $PN = \frac{b}{a}pN$. As NM is diminished the sum

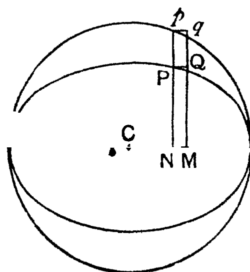


FIG. 81

of the rectangles PM approaches the area of the ellipse and the sum of the rectangles pM approaches the area of the

circle. Hence, in the limit, the area of the ellipse is $\frac{b}{a}$ times the area of the circle $=\frac{b}{a}(\pi a^2)=\pi ab$.

EXAMPLE. To find the centre of the ellipse whose equation is $2x^2+4x+y^2-y=2$.

This must be written in the form $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$.

We have
$$x^2+2x+\frac{y^2-y}{2}=1$$
$$x^2+2x+1+\tfrac{1}{2}\ (y^2-y+\tfrac{1}{4})=1+1+\tfrac{1}{8}$$
$$(x+1)^2+\frac{(y-\frac{1}{2})^2}{2}=\frac{17}{8}$$
$$\therefore\ \frac{(x+1)^2}{(\frac{17}{8})}+\frac{(y-\frac{1}{2})^2}{(\frac{17}{4})}=1$$

The co-ordinates of the centre are therefore $-1, \frac{1}{2}$. If the origin were at the centre the equation would be
$$\frac{x^2}{(\frac{17}{8})}+\frac{y^2}{(\frac{17}{4})}=1$$

Here $\frac{17}{8}=b^2$ and $\frac{17}{4}=a^2$, therefore the major axis is parallel to the Y axis.

Its eccentricity is given by $b^2=a^2(1-e^2)$
$$\therefore\ \frac{17}{8}=\frac{17}{4}(1-e^2)$$
$$\therefore\ e=\frac{1}{\sqrt{2}}$$

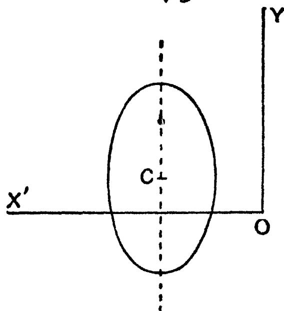


FIG. 82

EXAMPLES XV

1. Find the equation of the ellipse whose centre is at the origin, and for which $a=4$, $b=2$, when the major axis (i.) is along **OX**; (ii.) along **OY**.

2. Find the equation of the ellipse centre (2, 1); $a=4$, $b=2$, and the major axis parallel to **OX**. Find also the eccentricity of this ellipse.

3. Find the co-ordinates of the centre of the following ellipses: $9x^2-36x+4y^2+16y=0$; $4x^2+16x+9y^2-36y=0$; $x^2-2x+4y^2-16y-2=0$.

4. Find the co-ordinates of the foci of the ellipses given in Qn. 3.

5. Find the equation of the ellipse having a focus at (1, 2) directrix $3x+4y=5$, and eccentricity $\frac{1}{2}$.

6. Find the equation of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, if its centre were at (i.) 1, 2; (ii.) -2, -3, the axes of the curve being parallel to **OX** and **OY**.

7. Find the equations of the following ellipses if the centre were at the origin:—

(i.) $\frac{(x-2)^2}{4}+\frac{(y-9)^2}{9}=1$. (ii.) $x^2+2x+2y^2-6y=0$.

8. Given two fixed points **A** and **B**, where **AB**$=6$. Find in its simplest form the equation of the locus of a point **P** which moves so that **PA**$+$**PB**$=8$.

9. A circle radius r_1 is drawn inside a larger circle whose radius is r_2. A third circle is drawn touching both circles. Prove that the locus of its centre is an ellipse.

10. Find the eccentricity of the ellipse whose semi-axes are 4 and 3.

11. An ellipse is described by using an endless string which is passed over two pins. If the axes are to be 6 inches and 4 inches, find the necessary length of string and the distance between the pins.

12. Find the equation of the ellipse referred to its axes as axes of co-ordinates given (i.) that it passes through the points (2, 2) and (3, 1); (ii.) major axis 18, eccentricity $\frac{2}{3}$; (iii.) foci at (4, 0) and (−4, 0), eccentricity $\frac{1}{3}$; (iv.) distance between foci=8, minor axis 6.

13. A circle of radius 3 feet lying on an inclined plane is projected on to the horizontal plane. If the inclination of the plane is $\cos^{-1}\frac{2}{3}$, find the equation of the ellipse (origin at the centre) into which it projects, and find the area of the ellipse.

14. A vertical cylindrical pipe radius 6 inches passes through a roof which slopes at 60° to the horizontal. Find the area of the hole in the roof through which it passes.

15. Show from the relation $SP + S^1P = 2a$ that if the two foci coincide the ellipse becomes a circle. Hence show that the eccentricity of a circle is 0 and that the directrices are at an infinite distance from the centre.

16. Compare the areas of the triangles whose angular points are C (0, 0), P $(x_1 y_1)$, Q $(x_2 y_2)$, and C (0, 0), $p\left(x_1 \frac{by_1}{a}\right)$, $q\left(x_2 \frac{by_2}{a}\right)$. Hence show that if P and Q are two points on the auxiliary circle, and p, q the two corresponding points on the ellipse the area C$pq = \frac{b}{a}$ area CPQ.

17. Find the distance SB in an ellipse whose equation is $4x^2 + 9y^2 = 36$.

18. The major axis of an ellipse lies along OY and the centre is at (0, 1). If the semi-major axis is 2 and the eccentricity is $\frac{1}{\sqrt{2}}$, find the equation of the curve.

19. Two lines, A^1OA (6 inches) and B^1OB (4 inches), meet at right angles at O. On the edge of a narrow strip of paper take a point C and mark off along the edge CP=OA and CD =OA−OB. Place the strip so that C lies on B^1O and D on OA, and mark the locus of P. Suppose the strip to make an angle θ

with A^1OA and write down the co-ordinates of P referred to O as origin, and OA, OB as axes. Prove that P lies on an ellipse, and find its equation.

20. A point moves so that the sum of the squares of its distances from two given sides of an equilateral triangle is constant and equals $2c^2$. Prove the locus is an ellipse and find its eccentricity.

The Hyperbola.

Construction. When the eccentricity is greater than 1 the conic is a hyperbola.

Suppose $e=\frac{3}{2}$.

Take a point, S, 5 units from XD. Then A on SX 2 units from XD is on the curve.

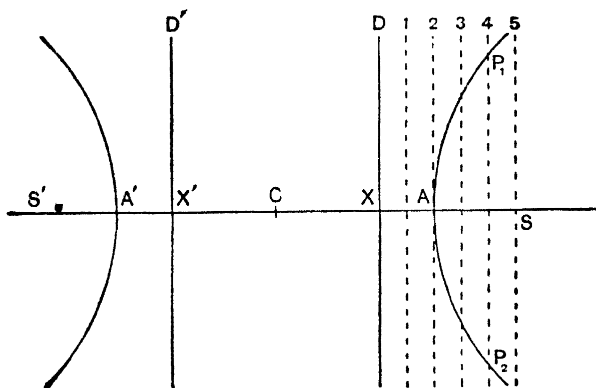


FIG. 83

With S as centre draw an arc of radius 6 units cutting the line 4 at P_1 P_2, and so on as in the ellipse.

Again if SA^1 be 15 units then $A^1X=10$ units, and

$\frac{SA^1}{A^1X}=\frac{3}{2}$ $\therefore$ A^1 is a point on the curve.

The curve has therefore two branches.

Let C be the mid-point of AA^1. Mark off $S^1A^1=SA$

and $A^1X^1 = AX$. Draw X^1D^1 parallel to XD. Then S^1 is a second focus and X^1D^1 the corresponding directrix.

Prove as for the ellipse that $CA = e\,CX$, and $CS = e\,CA = ea$.

Equation of the Curve.

Take C as origin, CS as X axis and a line at right angles to it through C as Y axis.

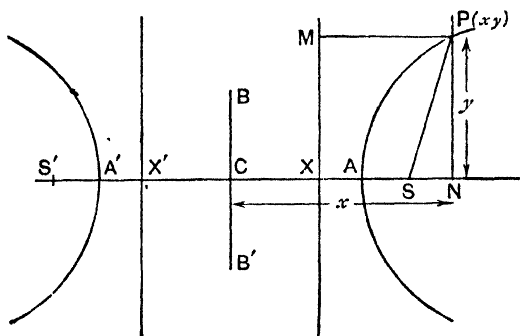


FIG. 84

Let $P\ (x, y)$ be any point on the curve and show as in the ellipse that

$$\frac{x^2}{a^2} - \frac{y^2}{a^2\,(e^2-1)} = 1$$

Now by analogy with the ellipse take $b^2 = a^2\,(e^2-1)$, then the equation reduces to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

When $x=0$, $y^2=-b^2$, which makes y imaginary, so that there is no point on the curve whose co-ordinates are o, b.

The line $AA^1 = 2a$ is called the transverse axis, and if $CB = CB^1 = b$ is marked off along CY, BB^1 is called the conjugate axis.

If $a=b$ the hyperbola is called a rectangular hyperbola.

Its equation is $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ or $x^2 - y^2 = a^2$

Change of Axes.

If the centre is at the point (α, β), show that the equation of the curve will be

$$\frac{(x-\alpha)^2}{a^2}-\frac{(y-\beta)^2}{b^2}=1$$

Mechanical Construction.

$$S^1P-SP=\text{constant.}$$

If P is any point on the curve

$$SP=e\ PM$$

$$S^1P=e\ PM^1$$

$$\therefore\ S^1P-SP=e\ (PM^1-PM)=e\ XX^1=2e\ CX=2a$$

$$\therefore\ S^1P-SP \text{ is constant.}$$

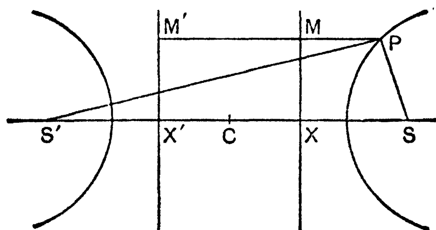


FIG. 85

Two strings, SPK, S^1PK, of unequal length, are fastened at S and S^1. KS is kept taut and rotated round S while a pencil at P keeps the shorter string in contact with the

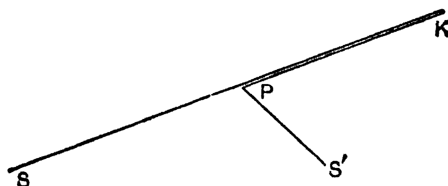


FIG. 86

longer at P. The locus of P is a hyperbola, for

$$SP-S^1P=SPK-S^1PK=\text{constant}$$

EXAMPLES XVI

1. Find the co-ordinates of the foci of the hyperbola $2x^2-3y^2=5$.

2. Find the length of the transverse axis, the co-ordinates of the centre, and the value of the eccentricity of the following hyperbolas:—(i.) $\frac{(x-1)^2}{9}-\frac{(y+\frac{1}{2})^2}{4}=1$; (ii.) $4x^2-y^2+16x+2y=13$.

3. Find the equation of the hyperbola having its focus at $(-1, 1)$, corresponding directrix the line $x+y-2=0$ and eccentricity $\sqrt{3}$.

4. Two fixed circles external to one another are touched by a third movable circle. Prove that the locus of the centre of this circle is a hyperbola.

5. A gun is fired from a point **A**. The sound reaches a point **B** at 1 o'clock, a second point, **C**, 2 seconds later, and a third point, **D**, 3 seconds after **C**. No three of these points are in a straight line. The velocity of sound being constant, prove that **A** lies at the intersection of two hyperbolas.

6. Find the eccentricity of the following hyperbolas:—(i.) $\frac{x^2}{5}-\frac{y^2}{4}=1$; (ii.) $\frac{y^2}{16}-\frac{x^2}{9}=1$. Trace the curves roughly.

7. Find the equations to the following hyperbolas referred to their axes as axes of co-ordinates:—(i.) given $a=3$, $b=2$; (ii.) given that the conjugate axis is 5 and the distance between the foci is 13; (iii.) distance between the foci $=12$ and eccentricity$=\frac{3}{2}$.

8. Find the eccentricity of a rectangular hyperbola.

9. In a rectangular hyperbola prove that $SP.S^1P=CP^2$.

10. The report of a rifle and the sound of the bullet striking a target reach several men on the same level as the range at the same moment. Prove that all the men lie on the same hyperbola.

11. Distinguish between the following curves:—(i.) $x^2-y^2=0$; (ii.) $x^2+y^2=a^2$; (iii.) $x^2-y^2=a^2$.

12. **ABCDE** are points along a straight line. **AB**$=1$ inch, **BC**$=2$ inches, **CD**$=1$ inch. For any position of **E** draw a

circle with centre **A** and radius **BE** to meet a circle centre **D** and radius **CE** at **P**. Prove that the locus of **P** is a hyperbola.

13. If x is large show that the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ can be written in the form $y=\pm\left(\frac{bx}{a}-\frac{ab}{2x}\right)$ approximately and hence show that it approaches the position of two lines drawn through the origin.

14 Prove that the straight lines $\frac{x}{a}-\frac{y}{b}=m$ and $\frac{x}{a}+\frac{y}{b}=\frac{1}{m}$ always meet on a hyperbola.

15. **P** (x_1y_1) is any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. **PN** is drawn perpendicular to **AA**^{1}, prove that **AN.A**^{1}**N**$=x_1{}^2-a^2$ and $\frac{\mathbf{PN}^2}{\mathbf{AN.A^1N}}=\frac{b^2}{a^2}$.

Parameters.

It is often convenient to express the two co-ordinates of a point in terms of a single variable generally called a variable parameter. For instance, the co-ordinates of a point on the circle $x^2+y^2=a^2$ may be written $x_1=a\cos\theta$, $y_1=a\sin\theta$.

These points satisfy the equation of the circle, and instead of having two expressions x_1 and y_1 connected by the relation $x_1{}^2+y_1{}^2=a^2$ we have only one variable, θ.

Similarly the co-ordinates of a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may be written $x_1=a\cos\theta$, $y_1=b\sin\theta$. In this case θ is the eccentric angle of the point. (See p. 96.)

The co-ordinates of a point on the parabola $y^2=4ax$ may be written $x=at^2$, $y=2at$. We may give any values we please to t and obtain the corresponding values for x and y. The values so obtained will always be co-ordinates of points on the parabola as they always satisfy the relation $y^2=4ax$.

EXAMPLES XVII

1. Prove that if $x=a\cos\theta$, $y=a\sin\theta$, then the locus of the point (x, y) is a circle.

2. Find the Cartesian equations of the curves given by (i.) $x=1+4t$, $y=2+3t$, (ii.) $x=at^3$, $y=2at$; (iii.) $x=ct$, $y=\frac{c}{t}$.

3. Show that for all values of θ the point given by $x=a\cos\theta+b\sin\theta$, $y=a\sin\theta-b\cos\theta$ lies on a circle.

4. Prove that $x=t^2+1$, $y=t+1$ represents a parabola.

5. Prove that if $x=3+2\cos\theta$, $y=4+2\sin\theta$, the locus of **P** (x, y) is a circle. Find the position of the centre and the length of the radius.

6. Show that if $x=a\sec\theta$, $y=b\tan\theta$, the locus of (x, y) is a hyperbola.

7. Prove that if am^2, $2am$ are the co-ordinates of one end of a focal chord of a parabola the co-ordinates of the other are $\frac{a}{m^2}$, $\frac{-2a}{m}$.

8. If $x=t+\frac{1}{t}$ and $y=t-\frac{1}{t}$, find the equation of the locus of (x, y) as t varies.

9. If $x=\frac{3at}{1+t^3}$ $y=\frac{3at^2}{1+t^3}$, find the equation of the locus of (x, y) as t varies.

10. If $x=a\cos^3\theta$, $y=b\sin^3\theta$, find the equation of the locus of (x, y) as θ varies.

11. If $x=\frac{2^t+2^{-t}}{2}$ $y=\frac{2^t-2^{-t}}{2}$, show that the point (x, y) lies on a rectangular hyperbola.

12. **OA** (a) and **OB**(b) are two lines at right angles. A semicircle is described on **OA** as diameter, and through **B** is drawn a line **BC** parallel to **OA**. **OQ**, which makes an angle θ with **OA**, cuts **BC** in **R** and the arc of the circle in **Q**. **P** is the intersection of a line through **Q** parallel to **OA** and a line **RN** through **R** parallel to **OB**. Express the lengths of **ON** and **PN** in terms of a, b, θ. Calling **ON**, x, and **PN**, y, show, by eliminating θ, that the locus of **P** is the curve $y=\frac{abx}{x^2+b^2}$.

Further Problems on Loci.

The following examples will illustrate various methods of dealing with problems on loci.

1. **AP** and **BP** are two lines drawn through fixed points **A** and **B** such that the angle **APB** is constant. To find the locus of **P**.

Take **AB** as **X** axis and the perpendicular bisector as **Y** axis. Let **APB**$=\alpha$ and **AB**$=2a$.

Suppose the gradient of **AP** to be m and of **BP** m^1. Then the equation of **AP** is

$$y-0=m(x+a) \quad \text{(i.)}$$

and of **BP**

$$y-0=m^1(x-a) \quad \text{(ii.)}$$

Also

$$\tan\alpha=\frac{m^1-m}{1+mm^1}$$

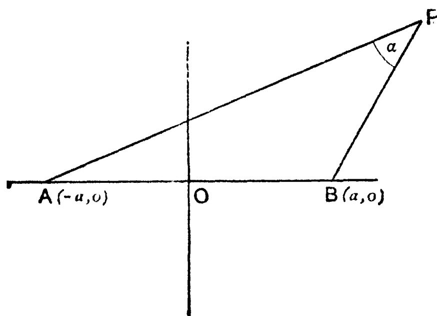


FIG. 87

If we eliminate m and m^1 between these equations we shall get a relation between x and y of the point **P**, since (i.) and (ii.) taken together determine the x and y of their point of intersection.

Now $m=\dfrac{y}{x+a}$ from (i.) and $m^1=\dfrac{y}{x-a}$ from (ii.).

$$\therefore \tan\alpha=\frac{y\left(\dfrac{1}{x-a}-\dfrac{1}{x+a}\right)}{1+\dfrac{y^2}{x^2-a^2}}=\frac{2ya}{x^2+y^2-a^2}$$

$$\therefore (x^2+y^2)\tan\alpha-2ay-a^2\tan\alpha=0$$

This is evidently the equation of a circle.

To find the centre, we have :

$$x^2+y^2-2ay \cot \alpha=a^2$$

i.e. $$x^2+y^2-2ay \cot \alpha+a^2 \cot^2 \alpha=a^2 (1+\cot^2 \alpha)$$

i.e. $$x^2+(y-a \cot \alpha)^2=a^2 \operatorname{cosec}^2 \alpha$$

$\therefore$ centre is $(0, a \cot \alpha)$ and $r=a \operatorname{cosec} \alpha$

Verify this by geometry.

2. A fixed point **A** inside a circle is joined to any point **B** on the circumference. Find the equation of the locus of the mid-point of **AB**.

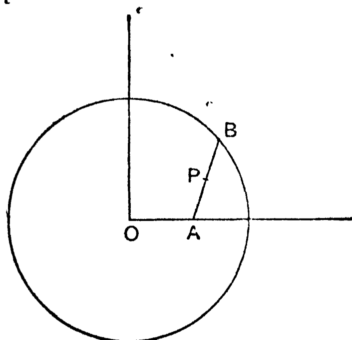


FIG. 88

In order to have the equation of the circle in the simplest form we take the origin at the centre. A further simplification is introduced by taking the line joining the centre to the point **A** as one of the axes.

We require the relation between the co-ordinates of the point **P**. Let **XY** be the co-ordinates of **P** and a the distance of **A** from **O**.

Since **P** is the mid-point of **AB**, we have, if x, y are the co-ordinates of **B** :

$$\mathbf{X}=\frac{a+x}{2} \qquad \mathbf{Y}=\frac{y}{2}$$

Since x, y is a point on the circle we know the relation between x and y; we therefore now express x, y in terms of **X** and **Y**.

$$x=2\mathbf{X}-a \qquad y=2\mathbf{Y}$$

But $x^2+y^2=r^2$ $\qquad \therefore (2\mathbf{X}-a)^2+(2\mathbf{Y})^2=r^2$

$\therefore 4\mathbf{X}^2+4\mathbf{Y}^2-4a\mathbf{X}+a^2-r^2=0$ is the equation of the required locus.

3. Two fixed lines at right angles to one another are cut by a third line of given length at **A** and **B**. Find the locus of the point **P** on **AB** which divides **AB** in the ratio **BP** : **PA**$=1:2$.

We take the two fixed lines as axes. It is required to find the relation between **XY** the co-ordinates of **P**.

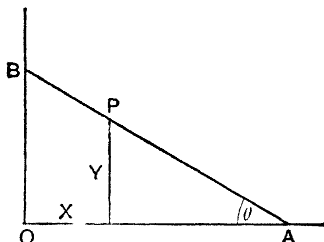


Fig. 89

Let **AB** make an angle θ with **OA**, and let l be its length, then **BP**$=\frac{l}{3}$ **PA**$=\frac{2l}{3}$.

From the figure we see $X=\frac{l}{3}\cos\theta$, $Y=\frac{2l}{3}\sin\theta$.

Since we know the relation between $\cos\theta$ and $\sin\theta$ we now express $\cos\theta$ and $\sin\theta$ in terms of **X** and **Y**.

$$\cos\theta=\frac{3X}{l} \qquad \sin\theta=\frac{3Y}{2l}$$

But $\cos^2\theta+\sin^2\theta=1$ $\qquad \therefore \frac{9X^2}{l^2}+\frac{9Y^2}{4l^2}=1$

which is the required equation.

Alternative Method.

Let the intercepts on the axes be **OA**$=a$, **OB**$=b$.

The co-ordinates of **P** dividing **BA** in the ratio 1 : 2 will be $X=\frac{a}{3}$, $Y=\frac{2b}{3}$.

If l be the length of AB we have $a^2+b^2=l^2$.

But $$a=3X,\ b=\frac{3Y}{2}$$

$$\therefore\ 9X^2+\frac{9Y^2}{4}=l^2$$ is the required equation.

EXAMPLES XVIII

1. Find the locus of the centre of a variable circle which touches a fixed line and a fixed circle.

2. Given the focus S of an ellipse and the corresponding vertex, show that the locus of the extremities of the minor axis is a parabola having its focus at S.

3. If the base of a triangle be given and the product of the tangents of its base angles, show that the vertex lies on an ellipse.

4. Any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is joined to B. Find the equation of the locus of the mid-point of BP.

5. If M be the mid-point of a chord, PP^1 of an ellipse, MT the perpendicular on a directrix, and S the corresponding focus, prove $SP+SP^1=2eMT$. Hence find the locus of the mid-point of a chord when the sum of the distances of its extremities from a focus is constant.

6. Find the locus of points which divide the ordinates of $y^2=4ax$ in the ratio $l:m$.

7. Find the locus of the mid-points of lines joining the vertex of a parabola to any point on the curve.

8. M is a fixed point in a fixed straight line KL. A circle of radius 1 inch rolls along KL and MP the tangent to it from M touches the circle at P. Find the equation of the locus of P.

9. A rod AB hinged at A rotates in a vertical plane. A cord fastened to B supports a weight W. Find the locus of W as the rod rotates.

10. Two concentric circles, radii r, R, are described about a fixed point C as centre, and any radius vector CPQ cuts them in P and Q. A rectangle, PAQB is described about PQ as

diagonal with **PA**, **PB** parallel to the axes. Prove that as the direction of **CPQ** varies the points **A** and **B** move upon ellipses, and find their equations.

11. A rope 30 feet long is hanging vertically from the roof of a gymnasium and just reaches the floor. A man puts his finger against the rope at a point 5 feet from the ground and walks forward in a straight line, keeping his finger always the same height above the ground. Assuming the upper part of the rope to keep straight, find the equation and eccentricity of the curve described by the free end of the rope.

12. Draw an isosceles triangle, **ABC**, having **AB**$=4$ cm., **BC**$=4$ cm. On **BC** take a point **P** and let **CP**$=a$ cm. The figure represents 2 rods **AB**, **BC**, smoothly jointed at **B** and lying on a table. The point **A** is fixed to the table and **C** is constrained to move along the line in which **AC** is lying at first. Show that as **C** moves towards **A** the point **P** describes a portion of an ellipse, and find the value of a if the eccentricity of the locus is 0·5.

13. Straight lines parallel to $y=kx$ are drawn to cut the 2 given lines $y=mx$ and $y=nx$. Find the equation of the locus of the mid-points of the parallel intercepts.

14. **A** (a, o) and **B** $(-a, o)$ are two fixed points; through them the lines **AP**, **BP** are drawn so that the gradient of **AP** $+$ the gradient of **BP** is constant. Find the locus of **P**.

15. Through two fixed points **A** (a, o) and **B** $(-a, o)$ lines **AP**, **BP** are drawn so that **AP**$=\lambda$**BP**, find the locus of **P**. Draw the locus when $\lambda=2$ and when $\lambda=1$.

CHAPTER VII

EQUATIONS OF TANGENTS TO THE CONIC SECTIONS

Circle.

We have seen on p. 35 that the gradient of the tangent to the circle $x^2+y^2=r^2$ is given by $\frac{dy}{dx}=-\frac{x}{y}$. At the point x_1y_1 the gradient is therefore $-\frac{x_1}{y_1}$.

The equation of the tangent at x_1y_1 is

$$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

i.e. $$yy_1+xx_1=x_1^2+y_1^2=r^2$$

or $$xx_1+yy_1=r^2$$

Tangent to $x^2+y^2+2gx+2fy+c=o$ at x_1y_1.

As on p. 35, we take a point P (x, y) on the curve and another point Q near P, whose co-ordinates are $x+\delta x$ and $y+\delta y$.

Then $(x+\delta x)^2+(y+\delta y)^2+2g(x+\delta x)+2f(y+\delta y)+c=0$.

Proceeding as before, we find

$$2x+\delta x+2y\frac{\delta y}{\delta x}+\left(\frac{\delta y}{\delta x}\cdot\right)\delta y+2g+2f\frac{\delta y}{\delta x}=0$$

As δx approaches zero, δy approaches zero and $\frac{\delta y}{\delta x}$. approaches its limiting value $\frac{dy}{dx}$.

$$\therefore \frac{dy}{dx}(y+f)=-(x+g) \qquad \therefore \frac{dy}{dx}=-\frac{x+g}{y+f}$$

At the point x_1y_1 the gradient is therefore $-\frac{x_1+g}{y_1+f}$ and the equation of the tangent is

$$y-y_1=-\frac{x_1+g}{y_1+f}(x-x_1)$$

or $$yy_1-y_1^2+yf-y_1f=-x_1x+x_1^2-gx+gx_1$$

i.e. $$x_1^2+y_1^2+gx_1+fy_1=xx_1+yy_1+gx+fy$$

But since x_1y_1 is on the curve

$$x_1^2+y_1^2+2gx_1+2fy_1+c=o$$

$$\therefore\ x_1^2+y_1^2+gx_1+fy_1=-gx_1-fy_1-c$$

Hence the equation of the tangent becomes

$$-gx_1-fy_1-c=xx_1+yy_1+gx+fy$$

or $$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=o$$

Otherwise. The gradient of the tangent may at once be obtained by using the differential notation.

Since $$x^2+y^2+2gx+2fy+c=0$$

$$\therefore\ 2xdx+2ydy+2gdx+2fdy=0$$

$$\therefore\ \frac{dy}{dx}=-\frac{x+g}{y+f}$$

N.B.—It will be found that for all *conics* the equation of the tangent at (x_1y_1) may at once be written down by writing xx_1 for x^2, yy_1, for y^2, $x+x_1$ for $2x$, $y+y_1$ for $2y$ and xy_1+yx_1 for $2xy$ in the equation of the curve.

Parabola. $y^2=4ax$.

As before, we take the points P (x, y) and Q $(x+\delta x, y+\delta y)$ on the curve and obtain $\frac{dy}{dx}=\frac{2a}{y}$.

The equation of the tangent at x_1y_1 is therefore

$$y-y_1=\frac{2a}{y_1}(x-x_1)$$

which reduces to $yy_1=2a(x+x_1)$.

Ellipse.

The gradient of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will be found to be $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$; and the equation of the tangent at x_1y_1:

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

reduces to
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

Hyperbola.

In a similar way the tangent to the hyperbola will be found to be
$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

Equations of Tangents in Terms of a Variable Parameter.

The gradient of the tangent when the co-ordinates of any point are given in terms of a variable such as t may easily be obtained by the relation $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$

or
$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

The gradient of the tangent to a circle at $x=a\cos\theta$, $y=a\sin\theta$ is
$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Now $\frac{dx}{d\theta}=-a\sin\theta$; $\frac{dy}{d\theta}=a\cos\theta$ $\quad\therefore \frac{dy}{dx}=-\cot\theta$

For the ellipse, show that the gradient at the point $x_1=a\cos\theta$, $y_1=b\sin\theta$, is $\frac{dy}{dx}=-\frac{b\cos\theta}{a\sin\theta}$.

For the parabola at $(at_1^2, 2at_1)$ show that $\frac{dy}{dx}=\frac{1}{t_1}$.

EXAMPLES XIX

1. Find the equations of the tangents to the circle $x^2+y^2=13$ at the points on the curve where $x=2$.

2. What is the equation of the tangent at the origin to the circle $x^2+y^2-5x+6y=0$.

3. Write down the equation of the tangent and the normal to the circle $x^2+y^2+3x-2y=0$ at $(-3, 2)$.

4. Find the equations of the tangents and normals to the following curves:—(i.) $y^2=6x$ at $\left(\frac{8}{3}, 4\right)$; (ii.) $\frac{x^2}{9}+\frac{y^2}{4}=1$ at (x_1y_1); (iii.) $x^2+5y^2=14$ at $(3, 1)$; (iv.) $x^2+x+y=3$ at $(1, 1)$.

5. Find the equation of the tangent to the curve $8y=x^3$ at the point x_1y_1.

6. Show that the line $y=2x$ meets the curve $2y=x^3$ in three points at two of which the tangents to the curve are parallel.

7. Prove that the parabolas $y^2=ax$ and $x^2=ay$ cut one another at an angle whose tangent is $\frac{3}{4}$.

8. The chain of a suspension bridge is in the form of a parabola. The span of the bridge is 200 feet and the dip of the chain is 50 feet. Find the length of the Latus rectum. Find the slope of the chain at its ends. If a tangent at the end cuts the roadway at P, find the distance of P from the middle of the bridge, the roadway being 6 feet below the lowest point of the chain.

9. From a fixed point $(h\ k)$ a line is drawn at right angles to the tangent to the parabola at P meeting a parallel to the axis through P in Q. Prove that the locus of Q for different positions of P is the curve $y\ (x+2a-h)=2ak$.

10. Find the gradient to the curve $x=at^3$, $y=2at$ at the point where $t=1$.

11. Find the equation of the tangent to the curve given by $x=t^2+2t-3$ and $y=2t^2$ at the point where $t=1$.

12. Find the gradient to $x=a\cos\theta+b\sin\theta$, $y=a\sin\theta-b\cos\theta$ where $\theta=\frac{\pi}{2}$.

13. The ordinate at P on an ellipse meets the auxiliary circle at p. Prove that the tangent to the ellipse at P and to the auxiliary circle at p meet the X axis at the same point.

Conditions for Tangency.

Note on the Roots of a Quadratic. The roots of the quadratic $ax^2+bx+c=0$ are

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Hence $\quad \alpha+\beta=\frac{-b}{a} \quad \alpha\beta=\frac{c}{a}$

The roots will be equal if $b^2-4ac=0$ and then $\alpha=\beta=-\frac{b}{2a}$.

If $c=0$ the quadratic reduces to $ax^2+bx=0$ or $x(ax+b)=0$. $\quad\therefore x=0$ or $\frac{-b}{a}$

Hence one of the roots is zero if $c=0$.

If also $b=0$ then the other root $\frac{-b}{a}$ becomes zero, so that two roots are zero if $c=0$ and $b=0$.

Let $x=\frac{1}{y}$, then the equation becomes $a+by+cy^2=0$.

If $a=0$ one of the values of y is zero.

$\therefore$ since $x=\frac{1}{y}$ one of the roots of x is infinite

If $a=0$ and $b=0$ two of the values of y will be zero and therefore two of the values of x will be infinite.

Condition that $y=mx+b$ should touch the circle $x^2+y^2=r^2$. If we are given the gradient m of a line it will be seen from a figure that two lines with this gradient will touch a given circle. It is required to find the value of b, which will make the line $y=mx+b$ (i.) a tangent to $x^2+y^2=r^2$ (ii.)

If the line cuts the circle, the two points of intersection will be found by solving the equations (i.) and (ii.). Thus x_1 and x_2 the roots of the equation

$$x^2+(mx+b)^2=r^2$$

or $$x^2(1+m^2)+2mbx+b^2-r^2=0$$

will give the abscissæ of the two points P_1 and P_2 common to both the line and the circle.

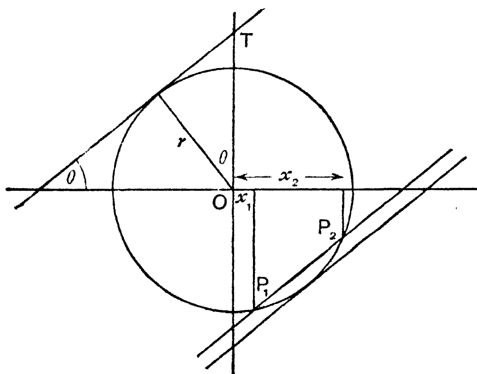


FIG. 90

If the line is a tangent, $x_1=x_2$. The condition for this is

$$4m^2b^2=4(1+m^2)(b^2-r^2)$$

from which $$b=\pm r\sqrt{1+m^2}$$

$\therefore$ the line whose equation is $y=mx\pm r\sqrt{1+m^2}$ will be a tangent to $x^2+y^2=r^2$.

If $m=\tan\theta$, $1+m^2=\sec^2\theta$, and $\therefore$ $b=\pm r\sec\theta$, which is clearly the case from a figure when the line is a tangent (OT, in Fig. 90).

Condition that $y=mx+b$ should touch the Parabola $y^2=4ax$. The points of intersection of the line and the curve are given by $$(mx+b)^2=4ax$$

or $$x^2m^2+x(2mb-4a)+b^2=0 \text{ (i.)}$$

The condition for equal roots is $4(mb-2a)^2=4m^2b^2$

i.e. $$b=\frac{a}{m}$$

$\therefore$ the line $y=mx+\dfrac{a}{m}$ is a tangent to the parabola $y^2=4ax$.

The point of contact is

$$x=-2\frac{(mb-2a)}{2m^2} \text{ from (i.)}$$

$$=-\frac{(a-2a)}{m^2}=\frac{a}{m^2}$$

Since $y^2=4ax$ we have $y^2=4a\dfrac{a}{m^2}$ $\qquad \therefore y=\dfrac{2a}{m}$

Hence when we take $x=at^2$, $y=2at$ as co-ordinates of a point on the parabola $y^2=4ax$, the t represents the co-tangent of the angle between the tangent to the curve and the X axis.

Condition that $y=mx+c$ should touch the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

N.B.—We take c for the intercept of the line to avoid confusion with the b of the ellipse.

A method similar to that described for the circle will give $c=\pm\sqrt{a^2m^2+b^2}$, and therefore the line $y=mx\pm\sqrt{a^2m^2+b^2}$ is a tangent.

If $b=a$ the ellipse becomes the circle whose radius is a and the condition reduces to $c=\pm a\sqrt{1+m^2}$ as before. (See p. 105.)

For the hyperbola we shall have $c=\pm\sqrt{a^2m^2-b^2}$, and the tangent is $y=mx\pm\sqrt{a^2m^2-b^2}$.

These equations for the tangents to conics are usefully employed when there is no necessity to introduce the co-ordinates of the point of contact of the tangents.

EXAMPLE. The locus of the point of intersection of two tangents to an ellipse at right angles to one another is a circle.

Let **X**, **Y** be the co-ordinates of the required point **P**. The line $y=mx+\sqrt{a^2m^2+b^2}$ is a tangent to the ellipse and since **P** (**X**, **Y**) is on it we have $\mathbf{Y}=m\mathbf{X}+\sqrt{a^2m^2+b^2}$ or $(\mathbf{Y}-m\mathbf{X})^2=a^2m^2+b^2$—*i.e.* $m^2(a^2-\mathbf{X}^2)+2\mathbf{YX}m+b^2-\mathbf{Y}^2=0$.

For any given position of **P** we have therefore a quadratic to determine m the gradient of the tangent through **P**. Since the two tangents are at right angles $m_1m_2=-1$, *i.e.* $\frac{b^2-\mathbf{Y}^2}{a^2-\mathbf{X}^2}=-1$ or $\mathbf{X}^2+\mathbf{Y}^2=a^2+b^2$.

P (**X**, **Y**) therefore lies on the curve $x^2+y^2=a^2+b^2$ which is a circle with its centre at the centre of the ellipse. It is known as the *Director Circle.*

EXAMPLES XX

1. Prove that $3x+4y=24$ touches the curve $xy=12$.

2. Find b if $4y=3x+4b$ is a tangent to $y^2=8x$.

3. Find the equations of the tangents to the following conics, which comply with the given conditions:—(i.) $y^2=4x$, gradient $\frac{1}{2}$; (ii.) $x^2+y^2=16$, gradient $-\frac{4}{3}$; (iii.) $x^2-4y^2=36$, perpendicular to $x-y+4=0$.

4. What is the condition for $4y-3x=4k$ to be a tangent to $x^2+y^2=r^2$?

5. Find the equations of the tangents to the ellipse $\frac{x^2}{2}+\frac{y^2}{7}=1$, which make 30° with **OY**.

6. Find the value of c if $x+y=c$ touches $2x^2+3y^2=4$.

7. Prove that $lx+my=1$ touches (i.) $y^2=4ax$ if $l=-am^2$; (ii.) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $a^2l^2+b^2m^2=1$.

8. If $x\cos 45°+y\sin 45°=p$ touches $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, find the value of p.

9. Prove that if the line $x\cos\alpha+y\sin\alpha=p$ touches the circle $(x-h)^2+(y-k)^2=r^2$ then $h\cos\alpha+k\sin\alpha-p=\pm r$, and explain the double sign.

10. A circle is described to pass through the origin and to touch the lines $x=1$, $x+y=2$, prove that the radius r is a root of the equation $r^2(3-2\sqrt{2})-2r\sqrt{2}+2=0$.

11. Find the points on an ellipse such that the tangents there are equally inclined to the axes. Prove that the length of the perpendicular from the centre on either is $\sqrt{\frac{a^2+b^2}{2}}$.

12. In an ellipse whose semi-axes are 4 and 3, find the co-ordinates of the points of contact of the tangents which make angles of 45° with the axes, and find the area of the square these tangents enclose.

13. Find the equation of the tangents to the circle $x^2+y^2-2x=3$ which are parallel to $3x+4y=0$.

14. Find the points on the parabola $y^2=4ax$ at which (i.) the tangent, (ii.) the normal is inclined at 30° to the axis.

15. Find the equations of the tangents to the parabola $y^2=9x$ which go through the point 4, 10.

16. Which of the straight lines $x=0$, $x=1$, $x=2$ is a tangent to $2x^2+3xy+y^2+x+y+1=0$? Ascertain whether the others cut the curve in real points.

17. Find the equations of the four common tangents to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

18. Show that the locus of the mid-points of the portions of the tangents to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cut off by the axes is $\frac{a^2}{x^2}-\frac{b^2}{y^2}=4$.

19. Find the co-ordinates of the point of intersection of tangents drawn to a parabola at the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$.

Locus of the Mid-Points of Parallel Chords.

When a straight line cuts a conic the equation giving the abscissæ of the points of intersection is a quadratic in x. If x_1 and x_2 are the two roots of this quadratic, X the abscissa of T, the mid-point of PQ, is equal to $\frac{x_1+x_2}{2}$. (Fig. 91.)

Similarly if y_1 and y_2 are the ordinates of P and Q, and Y the ordinate of T, $Y=\frac{y_1+y_2}{2}$.

Circle.

The quadratic whose roots give the abscissæ of the points of intersection of the line $y=mx+b$ and the circle $x^2+y^2=r^2$ is $x^2+(mx+b)^2=r^2$ or $x^2(1+m^2)+2mbx+b^2-r^2=0$

$$\therefore\ x_1+x_2=-\frac{2mb}{1+m^2}$$

$$\therefore\ X=\frac{x_1+x_2}{2}=-\frac{mb}{1+m^2}$$

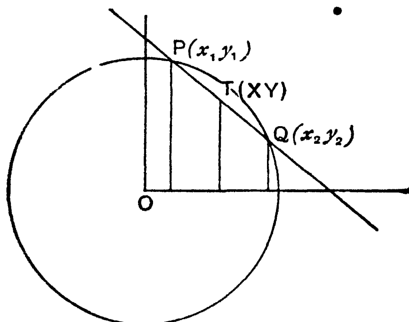


FIG. 91

Similarly the ordinates of P and Q are given by

$$\left(\frac{y-b}{m}\right)^2+y^2=r^2$$

or

$$y^2(1+m^2)-2by+b^2-m^2r^2=0$$

$$\therefore\ Y=\frac{y_1+y_2}{2}=\frac{b}{1+m^2}$$

$$\therefore\ \frac{Y}{X}=-\frac{1}{m} \qquad \therefore\ Y=-\frac{1}{m}X$$

Hence T lies on the line $y=-\frac{1}{m}x$, which is the equation of a line through O at right angles to $y=mx+b$.

Parabola.

The points common to $y=mx+b$ and $y^2=4ax$ are given by

$$y^2=4a\left(\frac{y-b}{m}\right)$$

or $$my^2-4ay+4ab=0$$

$$\therefore\ \text{Y}=\frac{y_1+y_2}{2}=\frac{2a}{m}$$

and since this is constant for a given value of m the locus of T is a line parallel to the axis of the parabola.

As Q approaches P the chord approaches the position of a tangent, which therefore touches the curve at a point whose ordinate is $\frac{2a}{m}$. Since this point is on the curve the abscissa is given by $\left(\frac{2a}{m}\right)^2=4ax$ or $x=\frac{a}{m^2}$.

The quadratic for x is $(mx+b)^2=4ax$

or $$m^2x^2+(2mb-4a)x+b^2=0$$

$\therefore\ \text{X}=\frac{x_1+x_2}{2}=\frac{2a-mb}{m^2}$, which depends upon the value of b.

Alternative Proof.

Let a secant drawn from a point P (X, Y) cut a curve in A and B, then if r be the distance PA, and θ the inclination of the secant to the X axis, the co-ordinates of A will be $\text{X}+r\cos\theta$, $\text{Y}+r\sin\theta$.

But this point is on the curve, so that if the curve is a parabola with equation $y^2=4ax$ we have

$$(\text{Y}+r\sin\theta)^2=4a(\text{X}+r\cos\theta)$$

or $$r^2\sin^2\theta+r(2\text{Y}\sin\theta-4a\cos\theta)+\text{Y}^2-4a\text{X}=0.$$

Now if B is the other point of intersection of the secant and the curve, the values of r (r_1 and r_2) given by this quadratic are the lengths of PA and PB.

If **P** is the mid-point of **AB** then $r_2 = -r_1$

$$\therefore r_1 + r_2 = 0$$

i.e. the sum of the roots of this quadratic equals zero

$$\therefore 2\mathrm{Y} \sin\theta - 4a \cos\theta = 0$$

or
$$\mathrm{Y} = 2a \cot\theta$$

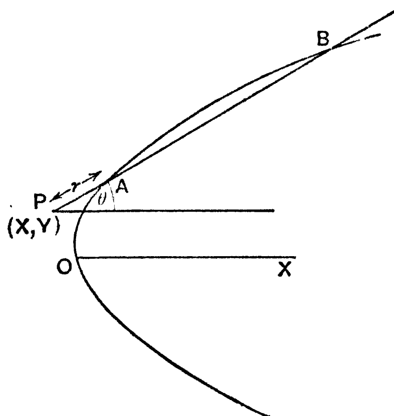


Fig. 92

Hence since θ is constant when parallel chords are drawn, the locus of the mid-points is $y = 2a \cot\theta$—*i.e.* a straight line parallel to **OX**.

Ellipse.

The points of intersection of $y = mx + c$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given by

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

or
$$x^2(b^2 + a^2m^2) + 2mca^2x + a^2(c^2 - b^2) = 0$$

$$\therefore \mathrm{X} = \frac{x_1 + x_2}{2} = -\frac{mca^2}{b^2 + a^2m^2}$$

Similarly for the ordinate

$$\frac{(y-c)^2}{m^2a^2}+\frac{y^2}{b^2}=1$$

or
$$y^2(b^2+a^2m^2)-2cb^2y+b^2(c^2-a^2m^2)=0$$

$$\therefore \ \mathsf{Y}=\frac{cb^2}{b^2+a^2m^2}$$

$$\therefore \ \frac{\mathsf{Y}}{\mathsf{X}}=-\frac{b^2}{ma^2}$$

The locus of T is therefore the line $y=-\frac{b^2}{a^2m^2}x$, a straight line through the origin.

If the gradient of this line is m^1 we have $m^1=\frac{-b^2}{a^2m}$ or $mm^1=-\frac{b^2}{a^2}$.

The locus of the mid-points of a series of parallel chords of a conic is called a *diameter* of the conic.

The diameter of the curve which passes through O (having a gradient m) and the locus of the mid-points of

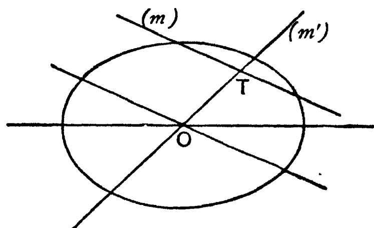


Fig. 93

chords parallel to this diameter (having a gradient m^1), are called Conjugate Diameters.

Hyperbola.

A similar investigation to the last for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ will give $\frac{\mathsf{Y}}{\mathsf{X}}=\frac{b^2}{ma^2}$, which is also a straight line through the origin.

Asymptotes.

The equation giving the points of intersection of $y=mx+c$ and the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

is
$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

or
$$x^2(b^2-a^2m^2)-2ma^2cx-a^2(c^2+b^2)=0.$$

If $b^2-a^2m^2=0$ one root of this equation is infinite (see p. 116). That is, any line whose gradient is $m=\pm\frac{b}{a}$ cuts the curve in one point at infinity.

On the tangent at A mark off a length b given by

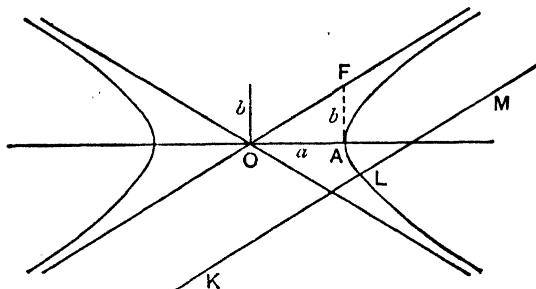


FIG. 94

$b^2=a^2(e^2-1)$. Then any line parallel to OF, such as KLM will only cut the curve in one finite point L.

If *also* $2ma^2c=0$—*i.e.* $c=0$ for $m\neq0$ and $a\neq0$—then two roots are infinite—*i.e.* the lines through the origin whose gradient is $+\frac{b}{a}$ or $-\frac{b}{a}$ will not cut the curve at any finite point.

These two lines are called asymptotes (ἀ συν πίπτω).

If $b=a$ each asymptote makes an angle of 45° with the axes, and they are therefore at right angles to one another. The equation of the curve is then $x^2-y^2=a^2$.

The value of e is given by $b^2=b^2(e^2-1)$ or $e=\sqrt{2}$. This hyperbola is called a Rectangular or Equilateral Hyperbola. (See p. 102.)

Equation of a Rectangular Hyperbola referred to its Asymptotes as Axes.

If the co-ordinates of P are x, y referred to the original

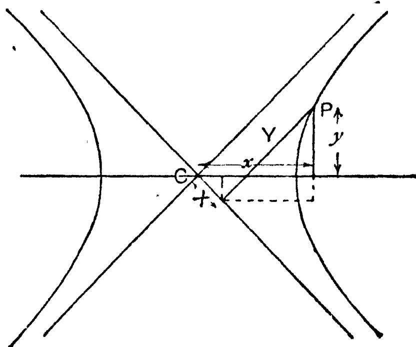


FIG. 95

axes, and X Y referred to the asymptotes as axes, we have

$$x = \mathrm{X} \cos 45° + \mathrm{Y} \cos 45°$$

$$y = \mathrm{Y} \sin 45° - \mathrm{X} \sin 45°$$

But
$$x^2 - y^2 = a^2 \text{ since } b^2 = a^2$$

$$\therefore \left(\frac{\mathrm{X}+\mathrm{Y}}{\sqrt{2}}\right)^2 - \left(\frac{\mathrm{Y}-\mathrm{X}}{\sqrt{2}}\right)^2 = a^2$$

or
$$\mathrm{XY} = \frac{a^2}{2}$$

Hence the equation of the curve is $xy = \frac{a^2}{2} = k$ a constant.

The co-ordinates of any point on this curve may be expressed in terms of one variable parameter by taking $x = \sqrt{k}t,\ y = \frac{\sqrt{k}}{t}$.

Equation of the Tangent to the Rectangular Hyperbola $xy=k$.

It may be shown as on page 34 that $\frac{dy}{dx}=-\frac{k}{x^2}=\frac{-y}{x}$.

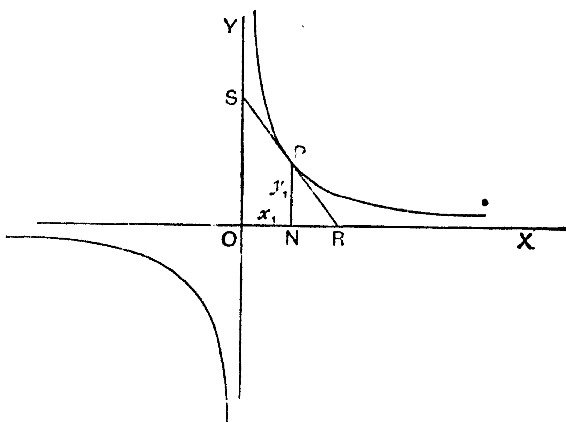


FIG. 96

At the point x_1y_1 the gradient is therefore $-\frac{y_1}{x_1}$, and the equation of the tangent is

$$y-y_1=-\frac{y_1}{x_1}(x-x_1)$$

or
$$x_1y-x_1y_1=-y_1x+y_1x_1$$

i.e.
$$\frac{x}{x_1}+\frac{y}{y_1}=2$$

This meets the axes where $x=2x_1$, and $y=2y_1$

$\therefore$ ON=NR $\qquad\therefore$ P is the mid-point of SR

Also the area of the triangle SOR$=\frac{1}{2}$ OS.OR

$$=\frac{1}{2}(2y_1)(2x_1)=2x_1y_1=2k=\text{constant}$$

If, then, we draw a series of lines SR such that $\triangle$ SOR =constant, these lines will touch a rectangular hyperbola.

EXAMPLES XXI

1. Find the equation of the locus of mid-points of chords to the parabola $y^2=4x$ which have a gradient of $\frac{1}{2}$.

2. Parallel chords having a gradient 1 are drawn in the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$. Find the locus of their mid-points.

3. Find the mid-points of the chords of $y^2=4x$ that lie on (i.) $y=5x-1$; (ii.) $y=5x-100$.

4. Find the locus of the mid-points of chords with gradient 5 for the curve $9x^2-y^2=36$.

5. Find the co-ordinates of the mid-point of the part of the line $y=3x$ that lies inside the circle $x^2+y^2+3x-6y-2=0$.

6. The line $3x+y=2$ meets the curve $4x^2+y^2=9$, find (i.) the middle point of the chord; (ii.) the equations of the tangents parallel to the chord.

7. Prove that the straight lines drawn from any point of a rectangular hyperbola to the extremities of any diameter are equally inclined to the asymptotes.

8. The ordinate through a point P of a hyperbola meets the asymptotes in Q and Q^1 and the hyperbola again in P^1. Show that (i.) $PQ.PQ^1=b^2$ and (ii.) $QP.QP^1=b^2$.

9. Prove that the eccentricity of a hyperbola equals the secant of the angle an asymptote makes with the X axis.

10. Find the equation of the rectangular hyperbola when the asymptotes intersect at $(\alpha\beta)$ instead of at the origin, the asymptotes remaining parallel to the axes of co-ordinates.

11. Find the co-ordinates of the point of intersection of the asymptotes of the hyperbola $xy-2x-3y+5=0$.

12. Find the equation of the tangent at (2, 1) to the hyperbola $xy=2$. If the axes are moved parallel to themselves so that the asymptotes meet at -1, 2, find the equation of the hyperbola and of the tangent when referred to the new axes.

13. A hyperbola has OY for one of its asymptotes and it cuts OX where $x=\pm 3$: it also passes through $(1, \frac{1}{2})$. Assuming its equation to be of the form $ax^2+2hxy+by^2=1$, find from the above information the values of b, a, and h. Plot the curve.

14. Find the equation of the tangent to the rectangular hyperbola $xy=c^2$ at the point $x=ct$, $y=\frac{c}{t}$.

15. Find the point of intersection of the tangents to $xy=c^2$ at the points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$.

16. Prove that the perpendicular SQ from the focus of a hyperbola to an asymptote is equal to b, and that CQ is half the transverse axis.

17. Show that the hyperbolas given by $4x^2-16y^2=k$ as k varies all have the same asymptotes.

18. PN is drawn from P (x, y) on a hyperbola parallel to one asymptote and cutting the other in N. If ON$=x_1$ NP$=y_1$ prove $x=(x_1+y_1)\cos\alpha$, $y=(y_1-x_1)\sin\alpha$ where $\tan\alpha=\frac{b}{a}$.

19. From a point P on a hyperbola a parallel to each asymptote is drawn to cut the other. If these points of intersection are M and N, prove that the rectangle PM, PN is constant.

20. Show that the equations of the two asymptotes of a hyperbola are given by $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$. Find the equation giving the abscissæ of the points of intersection of $y=mx+c$ and the asymptotes; hence find the locus of the mid-points of these segments for a series of parallel lines.

21. P and P^1 are two points on the same branch of a hyperbola and M is the mid-point of the chord PP^1. If the chord produced both ways meets the asymptotes at R and R^1, prove that M is the mid-point of RR^1 and hence that PR$=$P^{1}R^1. Hence show that if we are given the asymptotes and one point on the curve we may obtain any other points on the curve.

22. Prove that the product of the perpendiculars from any point on the hyperbola to the two asymptotes is constant.

23. Two branches of a hyperbola are shown on a diagram It is required to draw the axis. State the necessary construction.

24. Show that if a point moves to infinity on the parabola $y^2=4ax$ the gradient of the tangent approaches zero and the tangent approaches parallelism to the X axis, but if the curve is the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ the tangent approaches the position of an asymptote.

CHAPTER VIII

GEOMETRICAL PROPERTIES OF CONICS

The Parabola.

(i.) **AT=AN** **ST=SP** **PT** bisects $\widehat{SPM}$

The equation of the tangent at **P** (x_1y_1) to the parabola

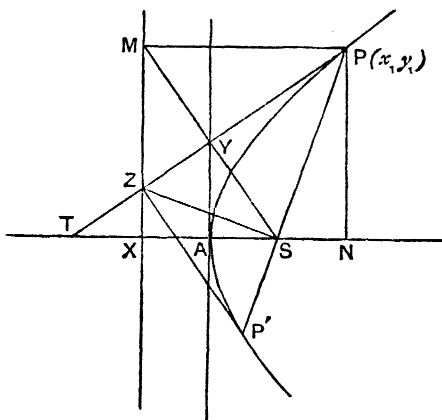


FIG. 97

$y^2=4ax$ is $yy_1=2a(x+x_1)$. At **T**, where this cuts the **X** axis, $y=0$ $\therefore x=-x_1$; *i.e.* **AT=AN.**

$$\therefore \text{AT}+\text{AS}=\text{AN}+\text{AX}$$

$$\therefore \text{ST}=\text{NX}=\text{PM}=\text{SP}$$

Hence it is easily proved by geometry that **PT** bisects the angle **SPM**.

(ii.) **SM** *meets the Tangent at P on the Tangent at the Vertex.*

Join **SM**, cutting **PT** at **Y**.

By geometry **SY**=**YM**, but **SA**=**AX** ∴ **AY** is parallel to **XM** and is therefore the tangent at the vertex.

By Analysis. The co-ordinates of **M** are $(-a, y_1)$. Write down the equations of **SM** and **PT** and show that these ines intersect where $x=0$.

(iii.) $SY^2=SP.SA$.

Since **TYS** is a right angle, **SY** is a mean proportional between **ST** and **SA**.

The Parabola as an Envelope.

If a point **S** be joined to a series of points Y_1, Y_2, Y_3 . . . on a straight line **AY** and perpendiculars Y_1P_1, Y_2P_2, etc.;

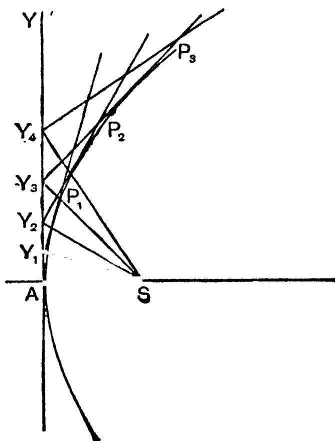


Fig. 98

be drawn to SY_1, SY_2, . . . respectively, it follows from Fig. 97, since **SYP** is a right angle, that Y_1P_1, Y_2P_2, etc. will be tangents to a parabola whose focus is **S**; **AY** being the tangent at the vertex.

The parabola so formed is called the Envelope of the lines Y_1P_1, etc.

(iv.) *If the tangent at* **P** *cut the directrix at* **Z**, **PSZ** *is a right angle.*

Prove geometrically that the triangles SPZ and ZPM are congruent. (Fig. 97.)

By Analysis. The line $y=mx+\frac{a}{m}$ is a tangent to the parabola and touches it at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

The equation of SP is $x\frac{2a}{m}+y\left(a-\frac{a}{m^2}\right)-\frac{2a^2}{m}=0.$

The equation of SZ is $y-0=\frac{m^2-1}{2m}(x-a).$

SZ meets the tangent where

$$\frac{m^2-1}{2m}(x-a)=mx+\frac{a}{m}$$

i.e. $$(m^2-1)x-a(m^2-1)=2m^2x+2a$$

or $$x(m^2+1)=-am^2+a-2a$$

$$=-a(m^2+1)$$

i.e. where $x=-a$

Hence SZ meets TP on the directrix.

(v.) Similarly if P[1] be the other extremity of the focal chord, SZ, at right angles to SP[1], will meet the tangent at P[1] on the directrix. *Hence tangents at the extremities of a focal chord meet at right angles on the directrix.* (Fig. 97.)

Otherwise. If P is (x_1y_1) the line joining S (a, o) to (x_1y_1) is $-xy_1+y(x_1-a)+ay_1=0$.

$$\therefore m=\frac{y_1}{x_1-a}$$

$$\therefore \text{ gradient of SZ is } -\frac{x_1-a}{y_1}$$

$$\therefore \text{ equation of SZ through } (a, o) \text{ is } y=-\frac{x_1-a}{y_1}(x-a)$$

This meets $yy_1=2a(x+x_1)$ where $-(x_1-a)(x-a)=2a(x+x_1)$ from which $x(a+x_1)=-a(a+x_1)$ or $x=-a$.

N.B.—This proposition is true of all conics.

(vi.) **SG=SP=ST.**

If the normal at P meets the axis of the curve at G then

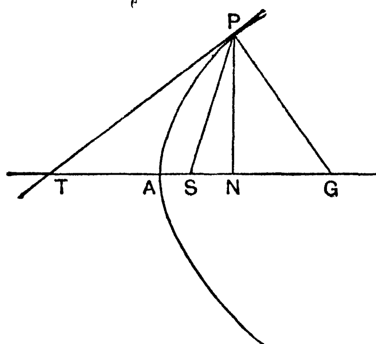


FIG. 99

a circle with centre S and radius ST goes through P and G, since TPG=90°   ∴ SG=SP=ST

(vii.) **The Subnormal is Constant.**

The equation of the tangent at P is $yy_1=2a(x+x_1)$.

∴ the equation of PG is

$$y-y_1=-\frac{y_1}{2a}(x-x_1)$$

At G where $y=0$, $2a=x-x_1$

∴ $x=x_1+2a$ ∴ AG=AN$+2a$

∴ NG$=2a$, which is a constant

NT is called the subtangent.

NG is called the subnormal.

(viii.) *The Curve whose Subnormal is Constant is a Parabola.*

For if PTA$=\theta$, $\tan\theta=\frac{dy}{dx}$

And NPG$=\theta$; PN$=y$

∴ NG$=y\tan\theta=y\frac{dy}{dx}=k$ (a constant)

∴ $\frac{y^2}{2}=kx+c$, which is the equation of a parabola

(ix.) Two Tangents from an External Point to a Parabola subtend equal Angles at the Foci.

Let P $(at_1^2, 2at_1)$, Q $(at_2^2, 2at_2)$ be two points on the parabola. The tangents at these two points intersect at T,

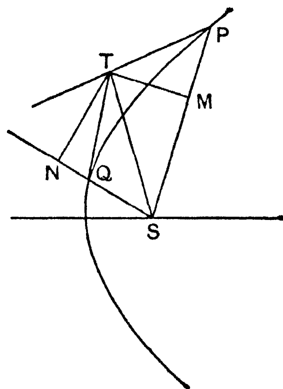


FIG. 100

whose co-ordinates are at_1t_2, $a\ (t_1+t_2)$. (See Example 19, p. 126.)

The equation of SP is $\dfrac{y}{2at_1}=\dfrac{x-a}{at_1^2-a}$

or
$$2t_1x-(t_1^2-1)\,y-2at_1=0$$

The length of the perpendicular from T on this is:

$$\frac{2at_1^2t_2-a\ (t_1^2-1)\ (t_1+t_2)-2at_1}{\sqrt{4t_1^2+(t_1^2-1)^2}}$$

$$=\frac{a\ (t_1^2+1)\ (t_2-t_1)}{t_1^2+1}=a\ (t_2-t_1)$$

Similarly the perpendicular TN on SQ is $a\ (t_1-t_2)$. Since these are equal in numerical value, T lies on the bisector of PSQ. Hence the tangents TP and TQ subtend equal angles at S.

This proposition is true of all conics.

EXAMPLES XXII

1. Prove that tangents to a parabola from a point on the directrix are perpendicular to each other.

2. Prove that tangents to a parabola at the extremities of the Latus rectum are perpendicular to one another.

3. Prove that the circle on a focal chord as diameter touches the directrix.

4. Find the locus of the foot of the perpendicular from the vertex to a normal.

5. The normal to the parabola $y^2=4ax$ at the point **P** meets the axis in **G**, and **GP** is produced outwards to **Q**, so that **PQ**=**PG**. Find the equation of the locus of **Q**.

6. **PNP**[1] is a chord of a parabola perpendicular to the axis **AN**. The normal at **P** cuts the axis in **G**. A circle described on **PP**[1] as diameter cuts at **Q**, **Q**[1] another chord of the parabola parallel to **PP**[1], which passes through **G**. Prove that for different positions of **PP**[1] the locus of **Q** is a parabola.

7. A tangent to a parabola at **P** cuts the axis at **T**. If **PN** is the ordinate of **P** prove that $\frac{PN^2}{TN}=2a$.

8. Plot the points (0, 0), (1, 0·5), (2, 2), (3, 4·5), (4, 8), (5, 12·5). Prove that a parabola with the **Y** axis for axis of the curve may be drawn through these points. Find its equation.

9. **PN** is an ordinate of a parabola. A straight line is drawn parallel to the axis to bisect **NP** and meet the curve in **Q**. Prove that **NQ** meets the tangent at the vertex in a point **T**, such that $AT=\frac{2}{3}NP$.

10. Prove that if the tangents to $y^2=4ax$ at (x_1y_1), (x_2y_2) meet at X_1Y_1, $X_1=\frac{y_1y_2}{4a}$ and $Y_1=\frac{y_1+y_2}{2}$.

11. Prove that the locus of the mid-points of that part of the normal between a parabola and the axis is a parabola whose vertex is the focus and **LR** $\frac{1}{4}$ of that of the original parabola.

12. **P** and **Q** are two points on a parabola whose vertex is **A** such that **AP** is perpendicular to **AQ**. **PH**, **QK** are the perpendiculars on the axis. Prove that **PH.QK** is equal to $16a^2$.

13. A line through the focus of the parabola $y^2=4ax$ cuts the curve at **P** (x_1y_1) and **Q** (x_2y_2). Prove that $x_1x_2=a^2$. If **AP**, **AQ** meet the Latin Rectum in P^1, Q^1 respectively, prove $SP^1=y_2$ and $SQ^1=y_1$.

14. A straight line through the focus of a parabola meets the curve in **P** and P^1. Show that the product of the ordinates of **P** and P^1 equals the square on the S.L.R.

15. The line drawn through the middle point **V** of a chord of a parabola perpendicular to the chord meets the axis in **G**, and the ordinate of **V** meets the curve in **P**. Prove that **PG** is the normal at **P**.

16. If two tangents, **TP**, **TQ**, are drawn to a parabola from **T**, prove $SP.SQ=ST^2$. Hence prove $S\hat{P}T=S\hat{T}Q$.

17. Take a point **S** 1 inch from a line **BC**. Join **S** to a series of points (**Y**) on **BC** and draw **YP** perpendicular to **SY**. Measure the Latus Rectum of the parabola these lines envelop.

Geometrical Properties of the Ellipse.

(i.) **PG**, *the normal at* **P**, *bisects* SPS^1.

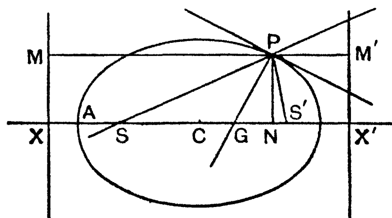


FIG. 101

$$SP=e\ PM=e\ (NC+CX)$$
$$=e\left(x_1+\frac{a}{e}\right)$$
$$=a+ex_1$$

Similarly $S^1P=a-ex_1$.

K

The equation of the tangent at P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

$\therefore$ the equation of PG is $y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$

This meets the X axis at G where $y=0$

$$\therefore\ -y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

$$\therefore \text{CG}=x=x_1-\frac{x_1b^2}{a^2}=\frac{x_1}{a^2}(a^2-b^2)=e^2x_1$$

$$\therefore \text{SG}=ae+e^2x_1,\ \text{also}\ \text{S}^1\text{G}=ae-e^2x_1$$

$$\therefore \frac{\text{SG}}{\text{S}^1\text{G}}=\frac{ae+e^2x_1}{ae-e^2x_1}=\frac{a+ex_1}{a-ex_1}$$

$$\therefore \frac{\text{SG}}{\text{S}^1\text{G}}=\frac{\text{SP}}{\text{S}^1\text{P}} \qquad \therefore \text{PG bisects SPS}^1$$

$\therefore$ the tangent bisects SPS^1 externally

(ii.) *If the tangent at* P *meets the directrix at* Z *the angle* PSZ *is a right angle.*

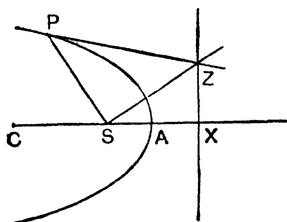


FIG. 102

The tangent at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$

This meets the directrix $x=\frac{a}{e}$ where $\frac{x_1}{ae}+\frac{yy_1}{b^2}=1$

or $$y=\frac{b^2(ae-x_1)}{aey_1}$$

The gradient of SZ is therefore

$$\frac{b^2(ae-x_1)}{aey_1}\div\left(\frac{a}{e}-ae\right)=\frac{b^2(ae-x_1)}{a^2y_1(1-e^2)}=\frac{ae-x_1}{y_1}$$

The gradient of SP is $\frac{y_1}{x_1-ae}$.

The product of these gradients is -1 ∴ PSZ is a right angle.

This proposition is true of all conics.

(iii). *If Perpendiculars* SZ, S^1Z^1 *be drawn from the Foci of an Ellipse to a Tangent then* $SZ.S^1Z^1=b^2$.

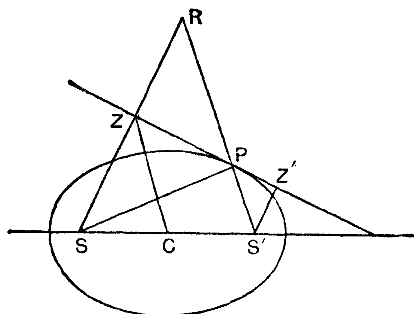


FIG. 103

The line $y=mx+\sqrt{a^2m^2+b^2}$ is a tangent to the ellipse.

SZ the perpendicular to it from S $(-ae, 0)$ is

$$\frac{+mae-\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}$$

Also $$S^1Z^1=\frac{-mae-\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}$$

$$\therefore\ SZ.S^1Z^1=\frac{-m^2a^2e^2+a^2m^2+b^2}{1+m^2}$$

$$=\frac{m^2a^2(1-e^2)+b^2}{1+m^2}$$

$$=\frac{m^2b^2+b^2}{1+m^2}=b^2$$

(iv.) *The Perpendicular drawn from a Focus of an Ellipse to a Tangent meets the Tangent on the Auxiliary Circle.*

SZ is drawn perpendicular to PZ, the tangent at P. Produce S^1P to meet SZ at R.

Since $\hat{SPZ} = \hat{ZPR}$ $\therefore$ SP = PR

$\therefore S^1R = S^1P + SP = 2a$

But Z bisects RS and C bisects SS^1

$\therefore$ CZ $= a$ $\therefore$ Z lies on the auxiliary circle

Conversely.

If a point S within a circle (not the centre) be joined to any point Z on the circumference, the line ZP, drawn at right angles to SZ, will touch an ellipse of which S is one of the foci.

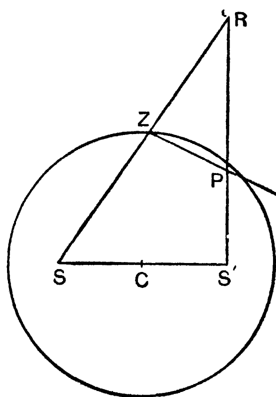


FIG. 104

Join S to C the centre and mark off $CS^1 = CS$. Produce SZ to R, making ZR = SZ, and join S^1R, cutting ZP in P.

Prove $SP + S^1P = 2CZ =$ constant.

Also prove that ZP bisects SPR and hence that ZP is a tangent to an ellipse at P.

If S be joined to several points Z on the circumference of the circle and perpendiculars such as ZP drawn to each, it will be seen that these lines envelop an ellipse.

EXAMPLES XXIII

1. In an ellipse prove that $AS.A^1S = b^2$.

2. PN is the ordinate of a point P on an ellipse. Prove that $AN.A^1N = a^2 - CN^2$. Hence show that

$$\frac{PN^2}{AN.A^1N} = \frac{b^2}{a^2}$$

3. Prove that if x_1 be the abscissa of a point P then $SP=a+ex_1$ and $S^1P=a-ex_1$.

4. A and B are two fixed points 4 inches apart and a point P is taken so that the sum of its distance from A and B is 5 inches. Taking as axes the line AB and its perpendicular bisector, find the equation of the locus of P. Find where the curve cuts each axis.

5. If any ordinate MP meets the tangent at the end of the L.R. in Q, prove that MQ=SP.

6. Find the equation referred to its axes of an ellipse whose eccentricity is $\frac{1}{3}$ and whose focus is 16 inches from the corresponding directrix. Prove that the line $2x\sqrt{2}+3y=24$ touches the curve and find the co-ordinates of the point of contact.

7. P is a point on an ellipse whose ordinate is y_1. Prove that the angle between the tangent at P and the focal distance of P is $\tan^{-1}\frac{b^2}{aey_1}$.

8. Prove that the sum of the squares of the perpendiculars on any tangent from two points on the minor axis each distant $\sqrt{a^2-b^2}$ from the centre is $2a^2$.

9. A sphere of radius 2 feet rests on a horizontal table, and a spot of light is held at a point 5 feet above the table and 4 feet measured horizontally behind the centre of the sphere. Find the length of the longest axis of the ellipse formed by the shadow of the sphere.

10. Prove that the perpendicular from the focus of an ellipse whose centre is C, on the tangent at any point P will meet CP on the directrix.

11. An ellipse slides between two straight lines at right angles to one another. Show that the locus of its centre is a circle. (See p. 119.)

12. If SY be drawn from the focus of an ellipse perpendicular to the tangent at P and YQ be the other tangent from Y to the ellipse, prove that SQ is parallel to S^1P.

13. The perpendicular from a focus to the tangent at an extremity of the Latus Rectum through the other focus meets the tangent on the minor axis.

14. Find the locus of the foot of the perpendicular from the centre to a tangent.

15. The tangent at a point meets the tangents at the extremities of the major axis at points K, K^1. Prove that KK^1 subtends a right angle at either focus. Also prove that $AK.A^1K^1=b^2$.

16. Draw a circle of radius 2 inches and take a point S, 1·5 inches from the centre. Join S to a number of points (Z) on the circumference. Draw lines (ZP) at right angles to each, and measure the axes of the ellipse these lines envelop.

The Hyperbola.

(i.) *The normal* PG *bisects the angle* SPS^1 *externally, and hence the tangent* PT *bisects* $S\hat{P}S^1$ *internally.*

The proof is similar to that given in the corresponding proposition for the ellipse.

FIG. 105

Find the co-ordinates of G and prove that

$$\frac{SG}{S^1G}=\frac{SP}{S^1P}$$

Auxiliary Circle.

The circle described on AA^1 as diameter is called the auxiliary circle.

The perpendicular from **S** to the tangent at **P** meets the tangent on the auxiliary circle.

Prove that $\mathbf{CZ}=\frac{\mathbf{S^1R}}{2}=\frac{\mathbf{S^1P-SP}}{2}=a$ as in the corresponding proposition for the ellipse.

Conversely.

If **S**, any point outside a circle, is joined to **Z** on the circumference, and **ZP** is drawn perpendicular to **SZ** prove that **ZP** is a tangent to a hyperbola having **S** as one of the foci.

Proof as in the ellipse.

(ii.) *The Portion of the Tangent to a Hyperbola intercepted by the Asymptotes is bisected at the Point of Contact.*

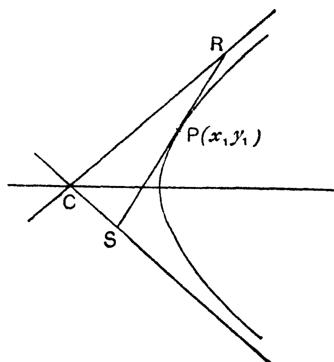


Fig. 106

The equation of the tangent at **P** (x_1y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

Where this meets the asymptotes whose equations are given by $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ we have $\frac{a^2}{x_1^2}\left(1+\frac{yy_1}{b^2}\right)^2-\frac{y^2}{b^2}=0$.

i.e.
$$y^2\left(\frac{y_1^2}{b^4}\frac{a^2}{x_1^2}-\frac{1}{b^2}\right)+2\frac{yy_1a^2}{x_1^2b^2}+\frac{a^2}{x_1^2}=0$$

This equation gives the ordinates of S and R.

If the roots are y_1 and y_2 the ordinate of the mid-point of SR is

$$\frac{y_1+y_2}{2} = -\frac{\dfrac{y_1a^2}{x_1^2b^2}}{\dfrac{a^2y_1^2-b^2x_1^2}{b^4x_1^2}}$$

$$= -\frac{y_1a^2b^2}{a^2y_1^2-b^2x_1^2} = \frac{-y_1}{\dfrac{y_1^2}{b^2}-\dfrac{x_1^2}{a^2}} = y_1$$

Hence P is the mid-point of RS.

(iii.) *If the tangent at* P *meets the directrix at* Z *the angle* PSZ *is a right angle.* Proof similar to that on p. 138.

EXAMPLES XXIV

1. P is any point on a rectangular hyperbola, C the centre, and A the vertex of the branch on which P lies, and PN the ordinate. Prove $AP^2=2CN\ AN$.

2. AB is the diameter of a circle on which P is any point. AP meets at Q the straight line drawn through B, such that BP.BQ are equally inclined to the tangent at B. Show that the locus of Q is a rectangular hyperbola.

3. In a rectangular hyperbola prove that $SP.S^1P=CP^2$.

4. In a rectangular hyperbola prove that if the normal at P meets the axes in G and g then $PG=Pg=PC$.

5. Prove that in an ellipse or a hyperbola CB is a mean proportional between the major axis and the latus rectum.

6. Prove that the straight line $y=mx$ bisects all chords of the hyperbola $xy=k^2$ which are parallel to $y=-mx$.

7. Find the locus of the mid-points of focal radii of a hyperbola.

8. If y_1y_2 are the ordinates of the points in which the tangent at x^1y^1 cuts the auxiliary circle, prove $\frac{1}{y_1}+\frac{1}{y_2}=\frac{2}{y^1}$.

9. The focus and directrix of a conic are given and also one point P on the conic. Show how to construct the tangent at P.

10. The triangle included between the asymptotes and any tangent is of constant area.

11. Prove that the asymptotes of a hyperbola meet the directrices on the auxiliary circle.

MISCELLANEOUS EXAMPLES

I

1. Find the equation of the circle $x^2+y^2=5$ if its centre were moved to the point $(-2, 3)$. Find the equation of the tangent to the circle in its new position at the point $(0, 4)$.

2. Find the equation of the parabola whose directrix is the **X** axis and whose focus is at $(0, 2a)$. What would be its equation if the origin were moved to the vertex, the axes remaining parallel to their original directions?

3. Rays of light from a certain point are all parallel after reflection at the surface of a parabolic reflector. State and prove the property of the parabola on which this fact depends.

4. Any point **P** on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is joined to **B** (o, b). Find the equation of the locus of the mid-point of **PB** and prove that it touches **OX**.

5. Show that the line $m^2x-my+a=0$ touches the parabola $y^2=4ax$.

J

1. Find the equation of the circle which has its centre at the point 3, 4 and touches the straight line $5x+12y=10$.

2. A tangent to the parabola $y^2=4x$ makes an angle of 60° with the axis. Find its equation and its point of contact.

3. Find the equations of the tangents to the circle $x^2+y^2=r^2$ which are perpendicular to the line $y=mx+b$.

4. A set square **ABC** with a right angle at **B** and **BC**$=2$ slides along the **X** axis, **BC** being along the axis and **C** towards the positive end; **C** is joined to the point **D** $(1, 1)$, cutting **AB** in **P**. Find the co-ordinates of **P** when **OB**$=t$ and deduce the equation of the path of **P**. Sketch the path of **P**, unit 1 inch.

5. Find the eccentricity of the conic

$$16x^2+36y^2-64x-36y-71=0$$

K

1. Prove that the circle described on the line joining the origin to $\left(c^3, \frac{1}{c^3}\right)$ as a diameter, passes through the point $\left(\frac{1}{c}, c\right)$. Hence justify the following construction for a cube root by means of the graph of $xy=1$. On **OX** lay off **OM**, whose cube root is required. Draw the ordinate **MP** to meet the curve in **P**. On **OP** as diameter describe a circle meeting the curve again in **Q**. From **Q** draw **QN** perpendicular to **OX**. Then **QN** is the required cube root.

2. Find the gradient of the tangent to the hyperbola $x^2-y^2=4$ at the point $\left(\frac{5}{2}, \frac{3}{2}\right)$. If **O** is the origin, **P** any point (x_1y_1) on the curve prove that the tangent at **P** makes the same angle with **OX** that **OP** makes with **OY**.

If this tangent meets the lines $y=x$ and $y=-x$ in **Q** and **R**, find the co-ordinates of **Q** and **R** and the area of the triangle **QOR**.

3. Draw the triangle **ABC**, **AB**$=7$ cms., **BC**$=10$ cms, **CA**$=12$ cms. Draw the two conics which pass through **A** and have **B** and **C** as foci. Find their eccentricities.

4. The focus and directrix of a conic are given and one point **P** on the conic. How would you construct the tangent at **P**? Prove the theorem you use if the conic is a parabola.

5. Take two points, **S**, **S**^{1}, 2 inches apart. Draw a line 1·5 inch from **S**, making an angle of 150° with **SS**^{1}, and intersecting **SS**^{1} beyond **S**^{1}. If an ellipse with **S** and **S**^{1} as foci touches the given line, find the point of contact and measure its distance from **S**. Find the eccentricity of the ellipse.

L

1. Draw a straight line **RYP** and mark off on it **RY**$=2$ cms., **YP**$=4$ cms. The line is a tangent to a parabola at **P** and it is cut by the tangent at the vertex in **Y** and by the directrix in **R**. Find by a geometrical construction the position of the focus, and give its distance from **P**. Find also the position of the directrix and give its distance from the focus.

2. The equation of a rectangular hyperbola is $xy=c$ referred to **OX**, **OY** as axes. If these axes are turned through an angle of 30° in the positive direction, find x and y, the old co-ordinates of any point **P** in terms of x_1y_1, the co-ordinates of **P** referred to the new position of the axes. Hence find the equation of the curve referred to the new position of the axes.

3. If the circle $2x^2+2y^2-x+7y-3=0$ cuts the axis of **Y** in **A** and **B**, find the length of **OC** when **O** is the origin and **C** the mid-point of **AB**.

4. A rod of length 3 inches slides with its ends on two fixed perpendicular wires, **OA** and **OB**. If **OP** is the perpendicular from **O** on the rod and θ the angle the rod makes with **OA**, prove that **OP**$=3 \sin \theta \cos \theta$. Find the equation of the locus of **P** referred to **OA**, **OB** as Cartesian axes of reference.

5. An ellipse is drawn with its major axis inclined at 45° to the horizontal. A horizontal line **AB** is a tangent to the upper end of the curve and **F** is the lower focus. **AB** represents a bar bearing on an elliptic cam which is free to rotate about **F**. If the equation of the cam is $\frac{x^2}{9}+\frac{y^2}{4}=1$, find the vertical height of the bar above **F** in the given position.

M

1. Draw a triangle **SPQ** having **SP**$=5$ cms., **SQ**$=10$ cms., **PQ**$=12$ cms. Consider **S** the focus of a conic of eccentricity 2; **P** and **Q** being two points on the curve. Construct the directrix corresponding to **S** and give its distance from **S**.

2. Find the (i.) co-ordinates of the centre; (ii.) co-ordinates of the foci; (iii.) length of major axis; (iv.) eccentricity of the ellipse $2x^2+5y^2=10x$.

3. Trace the curve $xy^2=4(2-x)$ and find its asymptote.

4. Prove that the equation $2x^2+3xy-2y^2+x+7y-3=0$ represents two straight lines, and find the angle between them.

5. Find the area of the triangle **OPQ** given **O**$(0, 0)$, **P**$\left(\frac{2^p+2^{-p}}{2}, \frac{2^p-2^{-p}}{2}\right)$, **Q**$\left(\frac{2^q+2^{-q}}{2}, \frac{2^q-2^{-q}}{2}\right)$. Find the locus of **P** for all values of p.

N

1. A point **P** moves so that the length of the tangent from **P** to the circle $x^2+y^2-2x-4y+1=0$ is equal to the ordinate of **P**. Prove that the locus of **P** is a parabola and find its equation.

2. **ABC** is an isosceles triangle whose base **BC** slides along **OX**, **B** starting from the origin. Prove that the intersection of the line joining **C** to the original position of **A** with the perpendicular **AM** on **BC** traces out a rectangular hyperbola.

3. Find the condition that the chord joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ on a parabola should pass through the focus.

4. A number of ellipses on the same major axis are cut by any common ordinate. Prove that the tangents at the intersections all meet in a point.

5. Take two points, **S** and **H**, 2 inches apart, and through **G** on **SH**, where **SG**$=1\cdot5$ inches, draw a line making an angle of $60°$ with **SH**. If this line is a normal to an ellipse whose foci are **S** and **H**, find by a geometrical construction the point **P** on the ellipse at which the line is normal. Measure **SP**.

O

1. The tangent to $y=x^2$ at the point where $x=t$ meets **OX** at **Q** and **R** is the mid-point of **PQ**. Find the co-ordinates of **R** in terms of t and prove that they satisfy the relation $9y=8x^2$.

2. Along **OX** mark off **OA**$=20$ and along **OY** mark **OB**$=10$. Complete the rectangle **OACB**. A point **P** travels uniformly from **B** to **C** and a point **Q** travels uniformly from **C** to **A**, the 2 points starting simultaneously from **B** and **C** and arriving simultaneously at **C** and **A**. Find the equations of **AP** and **BQ** when **Q** has travelled s cms. Find the equation of the locus of **R**, the point of intersection of **AP** and **BQ**.

3. If the co-ordinates of the foci of an ellipse are $(-c, 0)$ and $(c, 0)$, and the equation of a normal to the curve is $lx+my+n=o$, prove that the equation of the tangent at **P** to the ellipse is $\frac{x}{l}-\frac{y}{m}+\frac{c^2}{n}=0$.

4. Find the equation of the hyperbola whose focus is (2, 2), directrix $3x+4y=6$, and eccentricity 3. Find also the co-ordinates of its centre.

5. Show that $xy+3x+4y+4=0$ may be written in the form $(x+4)(y+3)=8$; hence find the equations of its asymptotes. What does the equation become if the origin is moved to the point (—4, —3), the axes remaining parallel to their original directions ?

P

1. The vertex **A** of an isosceles triangle is at (2, 1). The mid-point **B** of the opposite side is at (5, 7). The equal sides are inclined to **AB** at an angle of 45° on either side. Find their equations and the co-ordinates of the other two angular points.

2. Find what the equation $3y^2-2x^2-5xy+x-11y+6=0$ becomes when the origin is moved to (—1, 1), the axes remaining parallel to their original directions. Describe the locus of (x, y).

3. A circle cuts off a chord **AB** on the **X** axis whose mid-point is (p, o) and a chord **CD** on the **Y** axis whose mid-point is (o, q). If **OA.OB**$=c$ prove that the equation of the circle is $x^2-px+y^2-qy+c=0$.

4. The chain of a suspension bridge is a parabola and the roadway is supported by rods at intervals of 10 feet. If the span is 200 feet and the three middle rods are 5·4 feet, 5 feet, 5·4 feet in length, find the length of the next rods and the dip of the chain.

5. An archway is in the shape of a semi-ellipse, the road level being the major axis. If the breadth of the road is 30 feet and a man 6 feet high just touches the top when 2 feet from the side, find the greatest height of the arch.

Q

1. Describe a circle with centre **S** to touch a fixed line **AB** at **E**. On the other side of the line draw **CD** parallel to **AB** at a distance from it equal to **SE**. Take any point **Q** on **CD**. Join **QE** and produce it to cut the circle again in **K**. Produce **KS** to meet a parallel to **SE** through **Q** at **P**. Prove that **P** lies on a parabola whose focus is **S** and directrix **AB**.

2. A jet of water discharged from O hits a screen 6 feet away at a height of 4 feet above O, and when the screen is moved 4 feet farther away the jet hits it again at the same spot. Assuming the curve described to be a parabola, find the angle to the ground at which the jet is projected.

3. Tangents are drawn to an ellipse such that the product of their gradients is constant and equals λ. Show that the locus of their points of intersections is $y^2-b^2=\lambda(x^2-a^2)$.

4. A straight line AB of constant length slides between two axes, OX, OY. If the rectangle OAPB be completed and PQ be drawn perpendicular to AB to meet it in Q show that the locus of Q is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

5. If a normal to the rectangular hyperbola $xy=c^2$ at the point $\left(ct, \frac{c}{t}\right)$ meets the curve again at $\left(ct^1, \frac{c}{t^1}\right)$ prove that $t^3t^1=-1$.

CHAPTER IX

POLAR EQUATIONS

If a fixed point **O** is taken in a fixed line **OX**, and r (**OP**) is the distance of any point **P** from **O**, the position of **P** will be known when r and θ (the angle **POX**) are known. r and θ are called the polar co-ordinates of **P**; **O** is called the pole and **OP** the radius vector.

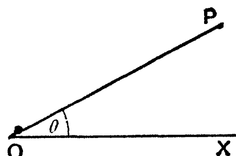


Fig. 107

If **OX** is the **X** axis and **OY** drawn at right angles to **OX** at **O** is the **Y** axis, then the Cartesian co-ordinates of **P** being (x, y) we have $x=r\cos\theta$, $y=r\sin\theta$.

By these relations we can at once change a Cartesian equation into polars: the equation of the circle $x^2+y^2=a^2$ becomes $r^2\cos^2\theta+r^2\sin^2\theta=a^2$ or $r^2(\cos^2\theta+\sin^2\theta)=a^2$—*i.e.* $r=a$.

To convert a polar equation into Cartesian we have

$$r=\sqrt{x^2+y^2} \text{ and } \theta=\tan^{-1}\frac{y}{x}.$$

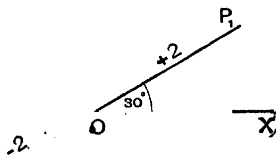


Fig. 108

Note.—If for a given value of θ, r is negative, the direction of **P** from the origin is negative—*e.g.* if when $\theta=30°$, $r=2$, the point is P_1, but if when $\theta=30°$, $r=-2$, the point is P_2.

Equation of a Circle in Polars.

The general equation of a circle in Cartesians is

$$x^2+y^2+2gx+2fy+c=0$$

Changing to polars we have:

$$r^2\cos^2\theta+r^2\sin^2\theta+2gr\cos\theta+2fr\sin\theta+c=0$$

or $$r^2+r(2g\cos\theta+2f\sin\theta)+c=0$$

If the circle passes through O and C its centre is on the initial line, the co-ordinates of the centre being $-g$ and $-f$, we have $-f=0$. Also since the circle passes through $(0, 0)$ we have $c=0$. If R is the radius of the circle, $-g=\mathrm{R}$.

∴ the equation becomes

$$r^2+r(-2\mathrm{R}\cos\theta)=0$$

or $$r=2\mathrm{R}\cos\theta$$

This equation can be at once obtained from a figure, for if $(r\ \theta)$ are the polar co-ordinates of P

$$r=\mathrm{OA}\cos\theta=2\mathrm{R}\cos\theta$$

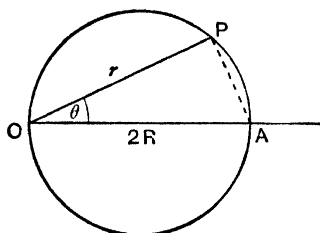


FIG. 109

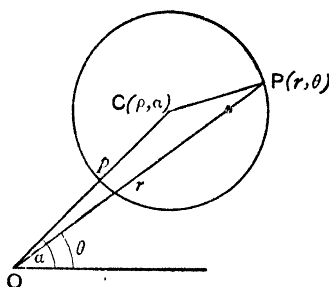


FIG. 110

The general equation may also be obtained directly, for if (ρ, α) are the polar co-ordinates of the centre

$$\mathrm{CP}^2=\rho^2+r^2-2\rho r\cos(\alpha-\theta)$$

or $$r^2-2r\rho\cos(\alpha-\theta)+(\rho^2-\mathrm{R}^2)=0$$

If $\rho=\mathrm{R}$ and $\alpha=0$ this reduces to $r^2-2r\mathrm{R}\cos\theta=0$ or $r=2\mathrm{R}\cos\theta$ as before.

Equation of a Straight Line in Polars.

The general equation in Cartesians is $ax+by+c=0$.

To transfer to polars we have $x=r\cos\theta$, $y=r\sin\theta$.

∴ the equation becomes

$$r(a\cos\theta+b\sin\theta)+c=0 \quad \text{(i.)}$$

Alternative Form.

The position of a line is fixed if we know the length of the perpendicular (p) to it from the origin and the angle (α) this perpendicular makes with the initial line.

If P (r θ) is any point on the line we see from the figure that

$p=r\cos(\alpha-\theta)$ (ii.) which is the polar equation of the line.

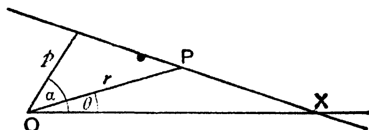


Fig. 111

This may be written $p=r\cos\alpha\cos\theta+r\sin\alpha\sin\theta$

or $$p=x\cos\alpha+y\sin\alpha$$

the perpendicular form of the equation in Cartesians.

It will be seen that (ii.) may be written in the same from as (i.)

Distance between Two Points in Polars.

If $(r_1\theta_1)$, $(r_2\theta_2)$ are the polar co-ordinates of A and B we have from Trigonometry

$$AB^2=r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)$$

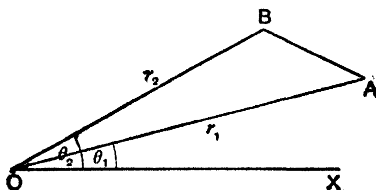


Fig. 112

Polar Equation of Conics.

The equations of the parabola, ellipse and hyperbola are most conveniently expressed in terms of their eccentricity and latus rectum.

Take S as the pole and let $SL=l$ be the S.L.R. of the conic.

FIG. 113

If $P(r\ \theta)$ be any point on the curve, and DX the directrix, we have

$$\begin{aligned}SP &= e\ PM\\ &= e(SN+SX)\\ &= e\ r\cos\theta + e\ SX \quad \text{but } SL=l=e\ SX\end{aligned}$$

$$\therefore\ r = e\ r\cos\theta + l$$

$\therefore\ l=r(1-e\cos\theta)$, which is the required equation

To Trace the Parabola.

$$2a=r(1-\cos\theta)=2r\sin^2\frac{\theta}{2}$$

i.e. $$r=a\ \text{cosec}^2\frac{\theta}{2}$$

We have

$\theta=0$	$\theta=\pm\frac{\pi}{3}$	$\theta=\pm\frac{\pi}{2}$	$\theta=\pm\frac{2\pi}{3}$	$\theta=\pm\pi$
$r=\infty$	$r=4a$	$r=2a$	$r=\frac{4a}{3}$	$r=a$

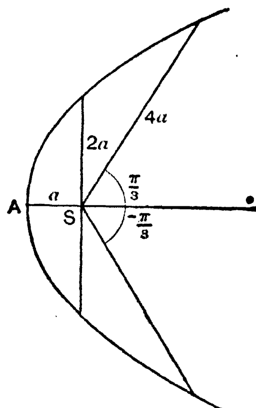


FIG. 114

To Trace the Hyperbola.

$$2=r(1-2\cos\theta) \text{ or } r=\frac{2}{1-2\cos\theta}$$

Substituting for θ we have the following values of r:—

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{8\pi}{9}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	-2	$-2{\cdot}7$	$-4{\cdot}8$	∞	$3{\cdot}1$	2	1	$\cdot8$	$\cdot7$	$\cdot67$

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	$\cdot7$	$\cdot8$	1	2	∞	$-4{\cdot}8$	$-2{\cdot}7$	-2

$\theta=0$ gives the point A^1 ($SA^1=-2$).

$\theta=\frac{\pi}{6}$ gives the point $r=-2{\cdot}7$, which is P^1 where $SP^1=2{\cdot}7$, since r is negative.

$\theta=\frac{\pi}{2}$ gives L.

$\theta=\pi$ gives A.

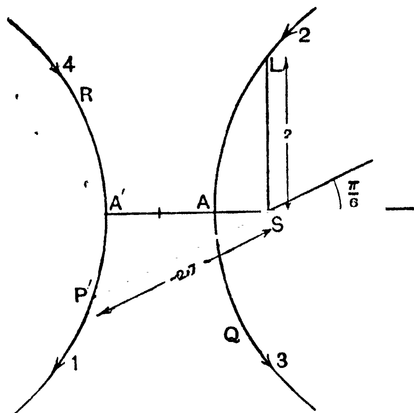


FIG. 115

We see that the part A^1P^1 is described first, then the right-hand branch LAQ, then the remainder, RA^1, of the left-hand branch.

The Curve $r^2=a^2\cos2\theta$. The Lemniscate.

Notes.

1. When $\theta=0$ or π we have $r=\pm a$, hence the curve crosses the initial line at points $\pm a$ from the pole.

2. Since the equation is unchanged if $-r$ is written for $+r$, the curve is symmetrical about the pole.

3. Since $\cos2(-\theta)=\cos2\theta$ the curve is symmetrical about the initial line.

4. No value of θ makes $r=\infty$.

5. r^2 is negative and r is therefore imaginary when $\cos2\theta$

is negative—*i.e.* when 2θ lies between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, and between $\frac{5\pi}{2}$ and $\frac{7\pi}{2}$.

$\therefore$ we need not calculate values of r for θ between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ or between $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

	o	15°	30°	45°
r	$\pm a$	$\pm \cdot 93a$	$\pm \cdot 71a$	o

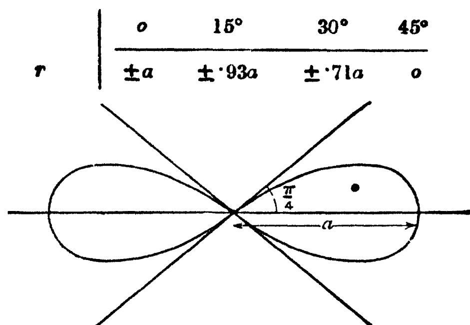


Fig. 116

The Limaçon $r=a\cos\theta+b$.

If OA$=a$ is the diameter of a circle and P$(r\ \theta)$ a point on the curve we see that OQ$=a\cos\theta$.

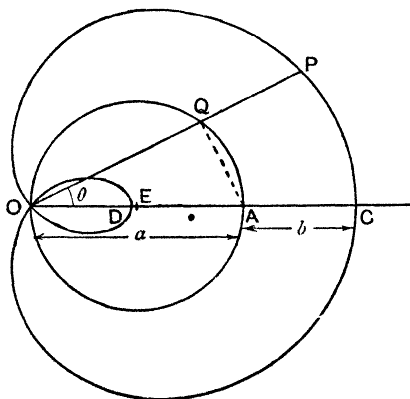


Fig. 117

If then O is joined to points on the circle and produced a distance b we get $r=OP=a\cos\theta+b$.

When $\theta=\frac{\pi}{2}$, $r=b$. When $\theta=\pi$, $r=-a+b$. If $a > b$ this distance is negative, and since when $\theta=\pi$ the positive direction of r is to the left, the corresponding point D will be to the right, where $OD=a-b$.

If $a=b$, D is at O and the loop disappears. The curve is then called a cardioid (heart-shaped).

EXAMPLE 1. Find the locus of the mid-points of the chords of the circle $r=2R\cos\theta$, which pass through the pole.

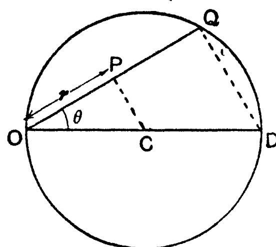


FIG. 118

The circle passes through the pole and has its centre C on the initial line.

Let P $(r\ \theta)$ be the mid-point of a chord OQ. Since PC is parallel to QD we see that $r=OC\cos\theta$ $=R\cos\theta$, which is the equation of the locus, clearly a circle on OC as diameter.

EXAMPLE 2. If a line is drawn from a point O outside a circle cutting it at P and Q, prove that OP.OQ is constant.

Taking the point O as pole and the equation of the circle as $r^2+r\,(2g\cos\theta+2f\sin\theta)+c=0$ we see that this is a quadratic for r whose roots $r_1 r_2$ will represent OP and OQ.

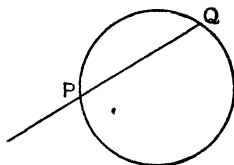


FIG. 119

Since the product of the roots$=c$, which is constant, we have $OP.OQ=$constant, the result being independent of θ.

The Cycloid.

The path of a point fixed on the circumference of a circle which rolls on a straight line is called a cycloid.

If P is the position of the point when the circle has turned through an angle θ radians from the position when P was

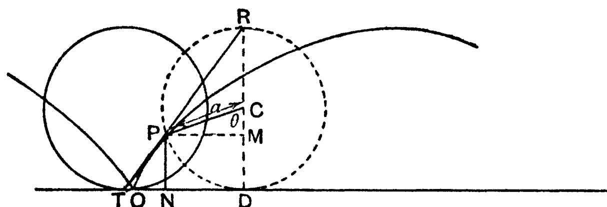


FIG. 120

at O, the co-ordinates (x,y) of P referred to O as origin will be $x=$ ON, $y=$ PN. If a is the radius of the circle

$$\text{arc PD}=a\theta=\text{OD}$$

Also $$\text{ND}=\text{PM}=a\sin\theta$$

and $$\text{CM}=a\cos\theta$$

$$\therefore\ x=\text{OD}-\text{ND}=a\theta-a\sin\theta$$

$$y=\text{CD}-\text{CM}=a-a\cos\theta$$

The gradient of the tangent will be

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a\sin\theta}{a(1-\cos\theta)}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$=\cot\frac{\theta}{2}=\tan\left(90^\circ-\frac{\theta}{2}\right)$$

If DC is produced to R and RP joined, cutting the X axis at T, we see that PRC$=\frac{\theta}{2}$.

$$\therefore\ \text{RTD}=90^\circ-\frac{\theta}{2}$$

$\therefore$ RTD is the slope of the tangent at P.

$\therefore$ RT is the tangent to the curve.

EXAMPLES XXV

1. Plot the points whose polar co-ordinates are

$$\left(1, \frac{\pi}{4}\right), \left(-2, \frac{2\pi}{3}\right), \left(6, 0\right), \left(-2, -\frac{\pi}{2}\right)$$

2. Change the following equations into polars and plot the curves:—(i.) $x-2y=0$; (ii.) $y+4=0$; (iii.) $x^2+y^2=4$; (iv.) $x^2-y^2=2$; (v.) $xy=6$; (vi.) $x^2+y^2-2x=0$.

3. Change the following equations into Cartesians:—

(i.) $\theta=\alpha$; (ii.) $r^2=a^2\cos 2\theta$; (iii.) $\frac{l}{r}=2\cos\theta+\sin\theta$;

(iv.) $r=a\cos\theta+b\sin\theta$.

4. Plot the loci of the following equations:—(i.) $r=4\cos\theta$;

(ii.) $r\cos\theta=4$; (iii.) $r=\frac{4}{1-\cos\theta}$; (iv.) $r=2(1-\cos\theta)$;

(v.) $r\cos\theta=a\sin^2\theta$; (vi.) $r=a\sin 2\theta$; (vii.) $r=a\theta$;

(viii.) $r=\frac{2}{1+\tan\theta}$; (ix.) $r^2\sin 2\theta=4$; (x.) $r=6(1+\cos\theta)$.

5. Plot the curve $r=2b\cos\theta+b$, the Limaçon when $a=2b$. See Fig. 117, and prove that in this case $Q\hat{E}P=\frac{1}{3}P\hat{E}O$, so that the curve can be used to trisect an angle. It is called the "Trisectrix."

6. Chords passing through a fixed point on a circle are extended their own lengths. Find the locus of their extremities.

7. Find the locus of the mid-points of the lines drawn from a fixed point to a given circle.

8. Find the polar equations of (i.) a straight line parallel to the initial line and at a distance 2 from it; (ii.) a straight line perpendicular to the initial line at a distance 2 from the origin.

9. Trace the curve *log* $r=\theta$, θ being in radians. This curve is called a spiral.

10. A string whose end was originally at **A** is unwound from a circle so that it is always tangential to the circle. If **OA**$=a$,

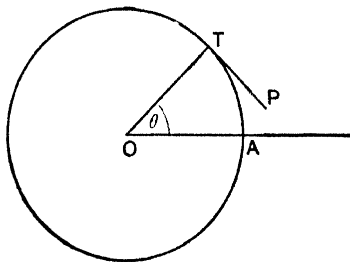


Fig. 121

find the Cartesian co-ordinates of **P** when **AOT**$=\theta$ radians. Plot the curve, and find the gradient in terms of θ.

11. Find the Cartesian equation of the curve $x=a\cos^2\theta$, $y=a\tan\theta$. Hence show that if a line is drawn through the origin to cut the circle $x^2+y^2=ax$ in **P** and the line $x=a$ in **Q**, the point which has the abscissa of **P** and the ordinate of **Q** is on the locus. Plot the locus by this method.

12. Show that the equation $r=\cos\theta+\sin\theta$ represents a circle. Find its radius and the position of its centre.

CHAPTER X

THREE DIMENSIONS

IF **ZOX**, **ZOY** represent two walls of a room at right angles to one another and **XOY** is the floor, the position of a point **P** in the room is determined by drawing **PN** perpendicular

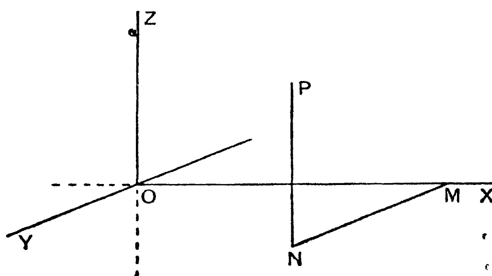


FIG. 122

to the floor. If we take **OX**, **OY** as axes, the co-ordinates of **N** will be **OM** (x) and **MN** (y). When we know **PN** (z) we have fixed the position of **P**.

The position of a point in space is thus fixed by knowing its 3 co-ordinates x, y, z referred to 3 axes mutually at right angles. The axis **OY** in the figure is at right angles to the plane of the paper and it must be remembered that **XOY**, **YOZ**, **OMN**, **MNP** are right angles.

The conventions used for signs are similar to those in two dimensions. If, for instance, **P** is behind **ZOX**, its y co-ordinate will be negative; if it is below **XOY** its z co-ordinate will be negative, and so on.

Distance between Two Points.

Let **P** $(x_1y_1z_1)$, **Q**$(x_2y_2z_2)$ be the two points where **OM**$=x_1$, **MN**$=y_1$, **NP**$=z_1$, etc.

Imagine **P** and **Q** to be the ends of the diagonal of a

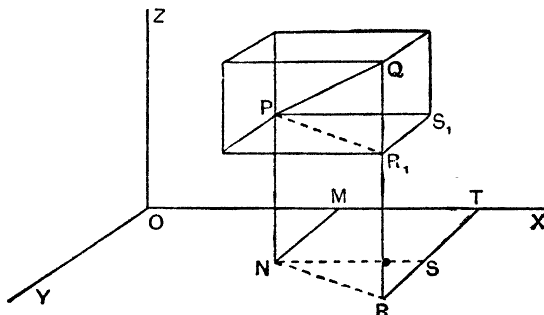


FIG. 123

rectangular box, with edges parallel to the axes, as in the figure.

If will be clear from a model that **PS**$_1=$**NS**$=$**MT**$=x_2-x_1$, and that **R**$_1$**S**$_1=$**RS**$=y_2-y_1$

Also **PS**$_1$**R**$_1$ is a right angle

$$\therefore \mathbf{PR}_1{}^2=(x_2-x_1)^2+(y_2-y_1)^2$$

But **QR**$_1=z_2-z_1$ and **PR**$_1$**Q** is a right angle

$$\therefore \mathbf{PQ}^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$$

The distance from $x_1y_1z_1$ to the origin is clearly given by

$$\mathbf{OP}=\sqrt{x_1{}^2+y_1{}^2+z_1{}^2}$$

Direction Cosines.

The direction of **OP** is determined by the angles (α, β, γ), called the direction angles, it makes with the axes.

From a model it will be clear that β is not an obtuse angle as it appears to be in the drawing. (Fig. 124.)

The cosines of the angles α, β, γ, are called the direction cosines of the lines, and we are more often concerned with the cosines of the angles than with the angles themselves.

If α and β are known, the position of **OP** is fixed and γ is therefore known, it must therefore be possible to find a

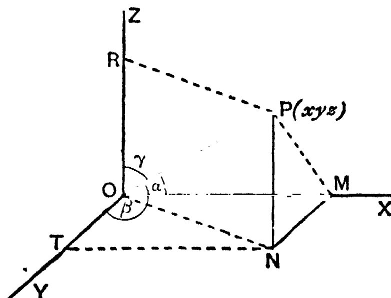


FIG. 124

relation between α, β, γ from which we can determine γ when we know α and β.

Let the co-ordinates of **P** be (x, y, z) and the direction angles of **OP** be α, β, γ.

Draw **PN** perpendicular to the plane **XOY**, from **N** draw **NM** perpendicular to **OX**. Join **PM**. The plane **PNM** is parallel to **ZOY** and **OM** is perpendicular to this plane, so **PMO** is a right angle.

$$\therefore \text{OM} = \text{OP} \cos \alpha \qquad i.e.\ x = \text{OP} \cos \alpha$$

If **PR** is drawn perpendicular to **OZ**, **OR**=**PN**, but

$$\text{OR} = \text{OP} \cos \gamma$$

$$\therefore z = \text{OP} \cos \gamma$$

If **NT** is drawn perpendicular to **OY** it can be seen from a model that **PTO** is a right angle and that **OT**=**OP** cos β $\therefore$ **MN**(y)=**OP** cos β.

The co-ordinates x, y, z are the projections of **OP** on the three axes.

Since $\text{OP}^2 = \text{ON}^2 + \text{NP}^2 = \text{OM}^2 + \text{MN}^2 + \text{NP}^2 = x^2 + y^2 + z^2$

and $x = \text{OP} \cos \alpha,\ y = \text{OP} \cos \beta,\ z = \text{OP} \cos \gamma$

we have $\text{OP}^2 = \text{OP}^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$

$$\therefore \cos^2\alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

This corresponds to the relation $\cos^2 \alpha+\sin^2\alpha=1$ in two dimensions, for if β is the angle **OP** makes with **OY**, $\beta=(90°-\alpha)$.

$\therefore\ \cos^2 \alpha+\cos^2 \beta=\cos^2 \alpha+\cos^2 (90°-\alpha)=\cos^2\alpha+\sin^2\alpha=1.$

Note that any line parallel to **OP** will have the same direction cosines as **OP**.

If we know that a, b, c are proportional to the direction cosines of a line we can easily determine the actual values of these cosines.

Let each of the equal ratios $\frac{\cos \alpha}{a}=\frac{\cos \beta}{b}=\frac{\cos \gamma}{c}$ be taken equal to k.

Then $\cos \alpha =ak$; $\cos \beta=bk$; $\cos \gamma=ck$.

$$\therefore\ \cos^2 \alpha+\cos^2 \beta+\cos^2 \gamma=a^2k^2+b^2k^2+c^2k^2=1$$

$$\therefore\ k^2 (a^2+b^2+c^2)=1 \qquad \therefore\ k=\frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore\ \cos \alpha=\frac{a}{\sqrt{a^2+b^2+c^2}}; \qquad \cos \beta=\frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$\cos \gamma=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

EXAMPLES XXVI

1. Find the distance from the origin to the points **P** (1, 3, 4) and **Q** (−1, −2, 4).

2. In Fig. 124, if the co-ordinates of **P** are 2, 3, 4, find the lengths of **NM**, **PM**, **ON**.

3. Find the distance between **P** and **Q**, given (i.) **P** (3, 1, 4), **Q** (2, 2, 0); (ii.) **P** (−3, 1, −2), **Q** (−2, −5, 1); (iii.) **P** $(1, \frac{1}{2}, 0)$, **Q** $(-\frac{1}{2}, 0, 1)$.

4. Describe the position of the line for which $\alpha=\beta=\gamma$, and find the values of these angles.

5. If $\alpha=45°$, $\beta=60°$, find γ when acute.

6. If $\alpha=60°$, $\beta=90°$, $\gamma=30°$, describe the position of the line.

7. The co-ordinates of **P** are 1, -2, 4, find α, β and γ.

8. The direction cosines of a line are proportional to 1, -2, 2. Find their true values.

9. A line through **O** is drawn in the plane **XOY**, making 30° with **OX** and another line in the plane **XOZ**, making 45° with **OX**. Write down the 3 direction cosines of each of these two lines.

10. The edges of a rectangular box are **OX**$=5$ inches, **OY**$=4$ inches, **OZ**$=6$ inches. Find the direction cosines of the diagonal of the face **YOZ** and of the diagonal of the box drawn through **O**.

Projection.

If from the extremities **A** and **B** of a line **AB** we draw perpendiculars **AM**, **BN** to a line **CD**, which **AB** does not

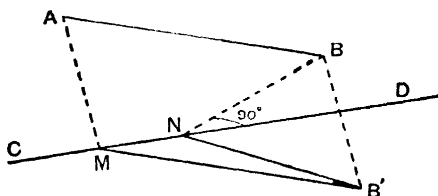


FIG. 125

intersect, then **MN** is the projection of **AB** on **CD**.

Through **M** draw **MB**^{1} parallel and equal to **AB**.

Then **AMB**^{1}**B** is a parallelogram. If through **BB**^{1} we draw a plane perpendicular to the line **CD**, cutting it in **N**, both **NB** and **NB**^{1} will be perpendicular to **CD**.

Now **AB** and **MB**^{1} being parallel make the same angle (α) with **CD**, and **MN**$=$**MB**^{1} $\cos\alpha=$**AB** $\cos\alpha$. The projection of a line **AB** on **CD** in three dimensions is therefore **AB** $\cos\alpha$ where α is the angle between the lines.

If **OA**, **AB**, **BC**, **CD** are projected on **OX** by drawing perpendiculars **AM**, **BN**, **CR**, **DS** to it, we see from the figure that the sum of the projections **OM**, **MN**, **NR**, **RS**$=$**OS**, which is the projection of **OD** on **OX**.

If **R** comes to the left of **N**, **RN** will be negative, and if we adopt the sign convention we shall see that in all cases it

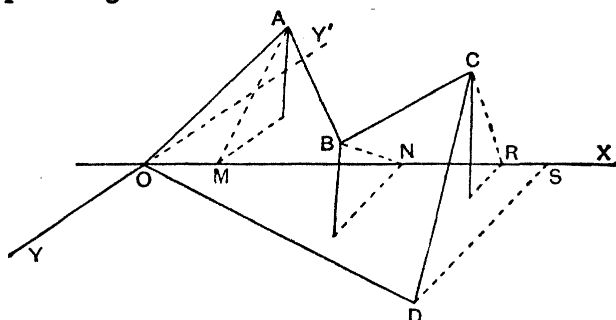


FIG. 126

is true to say that the algebraic sum of the projections on **OX** of any route from **O** to **D** equals the projection of **OD** on **OX**.

The Angle between Two Lines.

Let **OP**, $\mathbf{OP_1}$ be two lines through the origin parallel to the two given lines whose direction angles are α, β, γ,

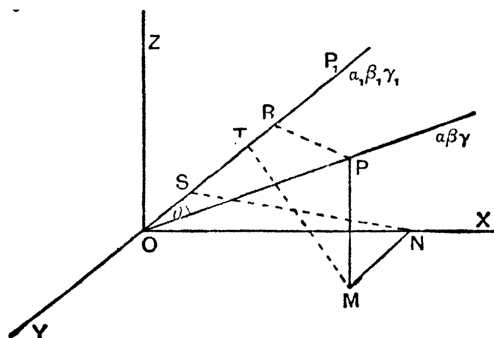


FIG. 127

and α_1, β_1, γ_1; it is required to find the angle θ between them.

Take a point **P** (x, y, z) on one of the lines (α, β, γ).

Project **OP** on to the other line by drawing **PR** perpendicular to it, then **OR**$=$**OP** $\cos\theta$.

Let a perpendicular from **N** on **OR** cut **OR** at **S**, and a perpendicular from **M** on **OR** cut **OR** at **T**.

Now the projection of **ON**, **NM**, **MP** on **OR** is
OS$+$**ST**$+$**TR**$=$**OR**$=$projection of **OP** on **OR**.

The angle between **OS** and **ON** is α_1

$$\therefore \text{OS}=x\cos\alpha_1$$

The angle between **ST** and **MN**$=$angle between **OR** and **OY**$=\beta_1$

$$\therefore \text{ST}=\text{MN}\cos\beta_1=y\cos\beta_1$$

The angle between **TR** and **MP**$=$angle between **OR** and **OZ**$=\gamma_1$.

$$\therefore \text{TR}=z\cos\gamma_1$$

$$\therefore \text{OR}=\text{OP}\cos\theta=x\cos\alpha_1+y\cos\beta_1+z\cos\gamma_1$$

$$\therefore \cos\theta=\frac{x}{\text{OP}}\cos\alpha_1+\frac{y}{\text{OP}}\cos\beta_1+\frac{z}{\text{OP}}\cos\gamma_1$$

But $x=\text{ON}=\text{OP}\cos\alpha$, $y=\text{OP}\cos\beta$, $z=\text{OP}\cos\gamma$

$$\therefore \cos\theta=\cos\alpha\cos\alpha_1+\cos\beta\cos\beta_1+\cos\gamma\cos\gamma_1$$

If $\theta=90°$, $\cos\alpha\cos\alpha_1+\cos\beta\cos\beta_1+\cos\gamma\cos\gamma_1=0$.

If $\gamma=\gamma_1=90°$ the two lines are in the same plane **XOY**, and the formula becomes $\cos\theta=\cos\alpha\cos\alpha_1+\cos\beta\cos\beta_1$.

i.e. $\cos\theta=\cos\alpha\cos\alpha_1+\sin\alpha\sin\alpha_1=\cos(\alpha-\alpha_1)$

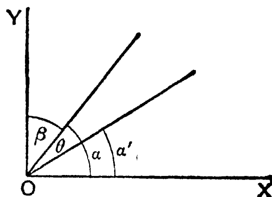


FIG. 128

Co-ordinates of a Point on the Line joining Two Points.

Given **P** $(x_1y_1z_1)$, **Q** $(x_2y_2z_2)$, it is required to find **(XYZ)** the co-ordinates of **R**, when $\frac{\text{PR}}{\text{RQ}}=\frac{l}{m}$.

If **MSN** is the projection of **PRQ** on the plane **XOY**, it is clear that $\frac{MS}{SN}=\frac{l}{m}$.

X, **Y** are therefore the same as in two dimensions:

$$X=\frac{lx_2+mx_1}{l+m} \qquad Y=\frac{ly_2+my_1}{l+m}$$

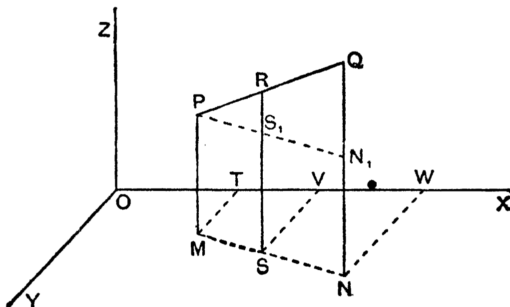


FIG. 129

If PS_1N_1 is drawn parallel to **MSN** we find from the similar triangles RS_1P and QN_1P that $Z=\frac{lz_2+mz_1}{l+m}$.

If **R** is the mid-point of **PQ** we see that

$$X=\frac{x_1+x_2}{2}; \quad Y=\frac{y_1+y_2}{2}; \quad Z=\frac{z_1+z_2}{2}$$

EXAMPLES XXVII

1. Find the direction cosines of the lines **OP**, **OQ**, given **P** (3, 1, 2), **Q** (1, −2, 1). Hence find the angle **POQ**.

2. The point **P** (3, 2, 4) is joined to the origin **O**. **PA**, **PB**, **PC** are drawn perpendicular to **OX**, **OY**, **OZ** respectively, and **PN** perpendicular to the plane **XOY**, **PM** perpendicular to the plane **ZOY**. Find the lengths of **OA**, **OB**, **OC**, **ON**, **OM**, **NM**.

3. From **P** (3, 1, 2) the perpendiculars **PA**, **PB**, **PC** are drawn to **OX**, **OY**, **OZ** respectively; **PN** to the plane **XOY**, **PM** to

M

the plane **YOZ**, and from **Q**$(-1, 2, 4)$ the corresponding lines **QA**^{1}, **QB**^{1}, **QC**^{1}, **QN**^{1}, **QM**^{1}. Find the lengths of **AA**^{1}, **BB**^{1}, **CC**^{1}, **NN**^{1}, **MM**^{1}, **NM**^{1}.

4. A line through **O** is drawn in the plane **XOY**, making 45° with **OX**, and another line in the plane **XOZ**, making 45° with **OX**. Find the angle between these two lines.

5. The edges of a rectangular box are **OX**$=5$ inches, **OY**$=6$ inches, **OZ**$=4$ inches. Find the direction cosines of the diagonal of the box through **O**, and the angle between it and the diagonal of the face **YOZ**.

6. At the two points **A** $(3, 2, 0)$, **B** $(5, 3, 0)$ lines **AP** and **BQ** are drawn perpendicular to the plane **XOY**. If **AP**$=4$ and **BQ**$=7$, find the co-ordinates of the mid-point of **PQ**.

7. Find the co-ordinates of the C. OF G. of the triangle **A** $(1, 3, 4)$, **B** $(5, -2, 1)$, **C** $(-2, 3, -4)$.

8. The co-ordinates of 3 of the angular points of a rhombus, **ABCD**, are **A** $(0, 0, 0)$, **B** $(3, -4, 4)$, **C** $(7, 1, 4)$. Find the co-ordinates of **D**.

9. Find formulæ for the co-ordinates of a point **T** on the line joining **P** $(x_1y_1z_1)$ and **Q** $(x_2y_2z_2)$, such that **T** lies on **PQ** produced, and divides it in the ratio **PT** : **TQ**$=l : m$.

10. The three corners of a pyramid are at the points **A** $(0, 0, 0)$, **B** $(4, 0, 0)$, **C** $(2, 5, 0)$. The vertex **D** is at 3, 3, 6. Find the direction cosines of **AD** and the angles **BAD**, **CAD**.

Equation of a Plane.

If p is the length of a perpendicular drawn from the origin to a plane, the locus of the plane is a tangent plane to a sphere of radius p. If the direction of p is known the position of the plane is fixed. We can therefore find the relation between x, y, z, the co-ordinates of any point on the plane, if we know p, the perpendicular from the origin to the plane, and α, β, γ its direction angles.

The method is the same as that used in finding the equation of a line in two dimensions when given p and α.

Let **P** (xyz) be any point on the plane. Draw **PR** per-

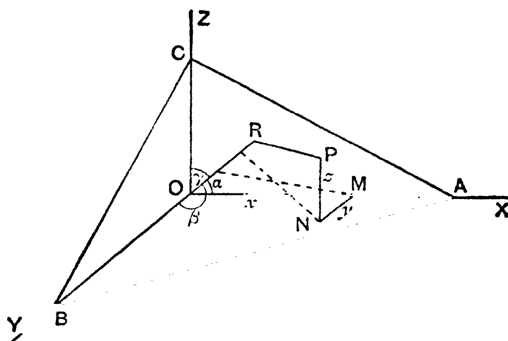


FIG. 130

pendicular to p (since **OR** is perpendicular to the plane, **PR** will lie on the plane).

Project **OM**, **MN**, **NP** on to **OR**.

Projection of **OM** is $x \cos \alpha$ (since α is the angle between **OX** and **OR**).

Similarly the projection of **NM** is $y \cos \beta$, and the projection of **NP** is $z \cos \gamma$.

$$\therefore x \cos \alpha + y \cos \beta + z \cos \gamma = \mathbf{OR} = p$$

The equation of the plane is therefore

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p$$

Equation of a Plane in the Intercept Form.

If the plane makes intercepts **OA** (a), **OB** (b), **OC** (c) on the axes, its equation in terms of these intercepts may be found from $x \cos \alpha + y \cos \beta + z \cos \gamma = p$.

If **R** is the foot of the perpendicular, **AR** will be perpendicular to **OR**

$$\therefore p = a \cos \alpha$$

Similarly $p = b \cos \beta$ and $p = c \cos \gamma$.

Since $$x\frac{\cos\alpha}{p}+y\frac{\cos\beta}{p}+z\frac{\cos\gamma}{p}=1$$
$$\therefore \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

N.B.—It will be noticed that the equation of the first degree in xyz represents a plane in 3 dimensions, whereas in 2 dimensions the equation of the first degree represents a straight line.

If we plot out all the points for which $z=0$ they will lie on the plane XOY. The equation of the plane XOY is therefore $z=0$. Similarly, the equation of XOZ is $y=0$ and of YOZ it is $x=0$.

If a plane is parallel to XOY the intercepts it makes on OX and OY are both infinitely large; the equation of the plane is therefore $\frac{x}{\infty}+\frac{y}{\infty}+\frac{z}{c}=1$—*i.e.* $z=c$.

It is clear that all the points for which z is a constant will lie on a plane parallel to XOY. Similarly a plane parallel to ZOY will have for its equation $x=a$ and if parallel to ZOX, $y=b$.

If a plane is parallel to the Z axis the intercept on this axis is infinite, and the equation becomes $\frac{x}{a}+\frac{y}{b}+\frac{z}{\infty}=1$

or $$\frac{x}{a}+\frac{y}{b}=1$$

An equation of the form $Ax+By=1$ in three dimensions therefore represents a plane parallel to OZ. Points on the plane $z=0$, which satisfy $Ax+By=1$, will lie on a straight line, and if z can have any value the locus is a plane through this line perpendicular to the plane $z=0$.

General Equation of a Plane.

The general equation of the first degree in xyz is $Ax+By+Cz=D$. We can divide through by one of these

coefficients (generally the constant term) so that we may take as our general equation $Ax+By+Cz=1$.

Comparing the equation $Ax+By+Cz=1$ with the equation $x\cos\alpha+y\cos\beta+z\cos\gamma=p$ we see that A, B, C must be proportional to $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

Let $$\frac{\cos\alpha}{A}=\frac{\cos\beta}{B}=\frac{\cos\gamma}{C}=k$$

then $$\cos\alpha=Ak,\ \cos\beta=Bk,\ \cos\gamma=Ck$$

but $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1 \qquad \therefore k^2(A^2+B^2+C^2)=1$

$$\therefore k=\frac{1}{\sqrt{A^2+B^2+C^2}} \qquad \therefore \cos\alpha=\frac{A}{\sqrt{A^2+B^2+C^2}}, \text{ etc.}$$

The equation $Ax+By+Cz=1$ written in the "perpendicular" form is therefore

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x+\frac{B}{\sqrt{A^2+B^2+C^2}}y+\frac{C}{\sqrt{A^2+B^2+C^2}}z$$
$$=\frac{1}{\sqrt{A^2+B^2+C^2}}$$

Note.—If $Ax+By+Cz=D$ and $A^1x+B^1y+C^1z=D^1$ are perpendicular, then $AA^1+BB^1+CC^1=0$. (See p. 168.)

If the plane goes through the origin it must be satisfied by (0, 0, 0); its equation is therefore of the form $Ax+By+Cz=0$.

If a plane is parallel to OZ we see that $\gamma=90° \ \therefore \cos\gamma=0$; the equation $x\cos\alpha+y\cos\beta+z\cos\gamma=p$ therefore becomes $x\cos\alpha+y\cos\beta=p$ or, in the general form, $Ax+By=1$.

If the plane contains the Z axis, it must be satisfied by any value of z, and the equation will be of the form $Ax+By=0$ since it also passes through the origin. In this case, if the equation is in the perpendicular form we note that p will be zero and since the direction of a perpendicular to the plane drawn from the origin will be perpendicular to OZ we have $\cos\gamma=0$: the equation therefore will be $x\cos\alpha+y\cos\beta=0$, which is the same form as $Ax+By=0$.

To write $\mathrm{A}x+\mathrm{B}y+\mathrm{C}z=1$ in the form $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ we can find where the plane cuts OX by putting $y=0$, $z=0$ in the equation; then $x=\frac{1}{\mathrm{A}}$, *i.e.* $a=\frac{1}{\mathrm{A}}$.

$$\therefore \text{ the equation is } \frac{x}{\frac{1}{\mathrm{A}}}+\frac{y}{\frac{1}{\mathrm{B}}}+\frac{z}{\frac{1}{\mathrm{C}}}=1$$

It is important to notice that when we compare $x \cos \alpha+y \cos \beta+z \cos \gamma=p$ with $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the direction cosines are not proportional to a, b, c, but to $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

Example 1. Find the equation of a plane making intercepts of 2, -1, 3 on the axes. Find the length of the perpendicular to this plane from the origin.

The equation of the plane in the intercept form is

$$\frac{x}{2}+\frac{y}{-1}+\frac{z}{3}=1$$

This becomes $3x-6y+2z=6$.

Multiplying throughout by a factor k to write it in the perpendicular form we have

$$3kx-6ky+2kz=6k$$

Comparing this with $x \cos \alpha+y \cos \beta+z \cos \gamma=p$, we have $\cos \alpha=3k$, $\cos \beta=-6k$, $\cos \gamma=2k$

$$\therefore\ 9k^2+36k^2+4k^2=1 \text{ or } k=\frac{1}{7}$$

The equation is therefore

$$\frac{3}{7}x-\frac{6}{7}y+\frac{2}{7}z=\frac{6}{7}$$

$$\therefore\ p=\frac{6}{7}$$

Example 2. To find the equation of the plane through the points (3, 5, 1), (2, 3, 0), (0, 13, 2).

The equation of any plane is $\mathrm{A}x+\mathrm{B}y+\mathrm{C}z=1$.

Since the point (3, 5, 1) is on it we have

$$3\mathbf{A}+5\mathbf{B}+\mathbf{C}=1$$

Similarly for the other points $\quad 2\mathbf{A}+3\mathbf{B}=1$

and $\quad 13\mathbf{B}+2\mathbf{C}=1$

$$\therefore\ \mathbf{A}=\frac{1}{4},\ \mathbf{B}=\frac{1}{6},\ \mathbf{C}=-\frac{7}{12}$$

The equation therefore is $3x+2y-7z=12$.

EXAMPLE 3. To find the area of the triangle joining the points (3, −3, 4), (4, 3, −2), (−2, 3, 10)

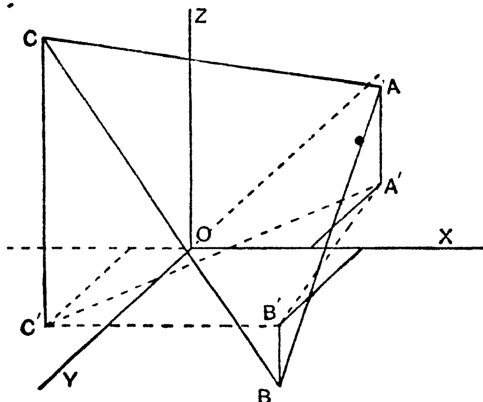


FIG. 131

Project the triangle on the plane $z=0$: the co-ordinates of the angular points will be $\mathbf{A}^1(3, -3)$, $\mathbf{B}^1$ (4, 3), $\mathbf{C}^1$ (−2, 3).

The area of the triangle $\mathbf{A}^1\mathbf{B}^1\mathbf{C}^1$ (see p. 72) will be 18 square units.

The equation of the plane **ABC** will be found to be $6x+2y+3z=24$. Writing this in the perpendicular form we have

$$\frac{6x}{7}+\frac{2y}{7}+\frac{3z}{7}=\frac{24}{7}$$

$\therefore$ the angle between the axis **OZ** and the perpendicular from the origin to the plane **ABC** is $\cos^{-1}\frac{3}{7}$, and this is the angle between the planes **ABC** and **XOY**.

$\therefore$ if Δ is the area of **ABC** we have by projection, $\Delta\times\frac{3}{7}=18$

$$\therefore\ \Delta=42 \text{ square units.}$$

EXAMPLES XXVIII

1. Find the equation of a plane which makes intercepts of -3, 2, 6 on the axes. Write the equation in the perpendicular form.

2. Find the equation of a plane which makes equal positive intercepts on the axes and passes through the point 1, 1, 1.

3. Find the equation of the plane passing through the points $(0, -1, -1)$, $(4, 5, 1)$, $(3, 9, 4)$.

4. Find the angle between the plane $6x-2y+3z=1$ and (i.) the plane **XOY**; (ii.) the plane **XOZ**; (iii.) the plane **YOZ**; (iv.) the axis **OX**.

5. Find the equation of a plane (i.) through the origin and the points $(1, 1, -1)$, $(2, 3, 4)$; (ii.) through the axis **OZ** and the point $(2, 1, 4)$.

6. Find the equation of a plane containing **OX** and the point $(2, 3, 1)$.

7. Find the length of the intercepts made on the axes by the plane $2x-3y+5z=10$.

8. Find the co-ordinates of the point in which the line $\frac{x}{2}=\frac{y}{1}=\frac{z}{3}$ meets the plane $x-2y+z=3$.

9. Find the length of the perpendicular from the origin on the plane $\frac{x}{2}+\frac{y}{1}-\frac{z}{5}=2$.

10. Find the angle between the two planes, $2x-y-2z=3$ and $3x+4y-6z=2$.

11. Find the equation of the plane parallel to **OY** whose intercepts on **OX**, **OZ** are 3 and -2.

12. Find the angle between the planes $6x-4y+3z=5$ and $z=3$.

13. Find the equation of a plane through $(-1, 0, 2)$, which is parallel to $x-2y+3z=4$.

14. The direction cosines of the perpendicular to a plane from the origin are proportional to 1, 3, 1, and the length of the perpendicular is 2. Find the equation of the plane.

15. The base of a triangular pyramid has its angular points at **A** (0, 1, 1), **B** (4, 2, 2), **C** (−3, −3, 2). The vertex is at **D** (1, 1, 5). Find the equations of the planes **ABD**, **ABC**, and the angle between them. Find the direction cosines of any line perpendicular to **ABC**.

16. Find the direction cosines of any line which is perpendicular to $2x-y-2z=3$.

17. **OA**, **OB** are the ground lines of two walls **COA** and **COB** at right angles to one another. A point **P** on **COA** is 5 feet from **OC** and 4 feet from the ground and a point **Q** on **BOC** is 4 feet from **OC** and 5 feet from the ground. Another point **R** on the ground is 4 feet from **OA** and 6 feet from **OB**. The corner is filled with earth so that the surface is plane and passes through **P**, **Q** and **R**. Find the angle it makes with the ground and the height it reaches above **O**. Find also its cubic contents.

18. A corner between two walls at right angles is required to be filled in symmetrically with earth so that the plane surface is 4 feet from the point **O** on the ground line where the walls meet. Find the distance from **O** to which the earth will extend along the wall.

19. Two lines, **OX**, **OY**, are marked on the ground parallel to the sides of a tennis court. A post whose top **P** is 5 feet above the ground is placed 6 feet behind **OX** and 8 feet behind **OY**. Another post, **Q**, 2 feet above the ground, is fixed 18 feet from **OY** and 2 feet behind **OX**. A bank of earth is to be made whose surface passes through **P** and **Q** such that the line along which it meets the level is parallel to **OX**. Find the depth of earth over **OX** and the slope of the surface.

20. A bench is 6 feet long and 4 feet wide. Taking the longer sides as **OX**, place an upright 2 feet high at (3, 2) and another 1 foot high at (4, 3). A sheet is stretched over these points and fixed along the edges of the bench. Find the equations of the six planes thus formed.

21. The plan of three walls, **BO**, **OA**, **AC**, has the angle at **O**$=90°$ and at **A**$=120°$. **OA**$=12$ feet, **OB**$=9$ feet and **OBC**$=90°$. **OACB** is covered by a plane roof resting on the

walls, 10 feet high at **A** and **B** and 8 feet high at **O**. Find the height of the roof at **C**. Taking **OA** as **X** axis, find the co-ordinates of the four corners of the roof and the equation of its plane.

22. Find the volume of the pyramid whose vertices are at **O** (0, 0, 0,) **A** (2, -3, 1), **B** $\left(0, 2, \frac{1}{3}\right)$, **C** (4, 0, -1), by finding the equation of the plane **ABC**, the area of the triangle **ABC** and the length of the perpendicular to it from the origin.

Equation of a Straight Line, given its Direction Cosines and one Point on it.

Let the direction angles of the line be α, β, γ, and **A** $(x_1y_1z_1)$ a point on it.

Let **P** (xyz) be any point on this line.

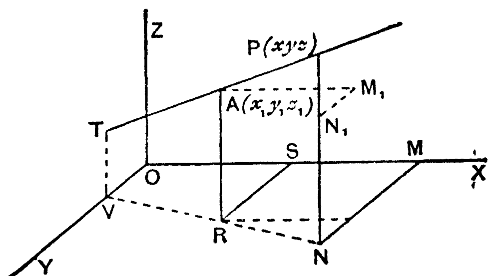


Fig. 132

Draw **AM**$_1$, **M**$_1$**N**$_1$ parallel respectively to **SM** and **MN**. Then **AM**$_1$$=x-x_1$, **M**$_1$**N**$_1$$=y-y_1$, **N**$_1$**P**$=z-z_1$.

Also since **MAP** is the angle α we have

$$\mathbf{AP}\cos\alpha=x-x_1$$

Similarly $$\mathbf{AP}\cos\beta=y-y_1$$

and $$\mathbf{AP}\cos\gamma=z-z_1$$

$$\therefore \frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\cos\beta}=\frac{z-z_1}{\cos\gamma}$$

which is the required equation.

If this plane cuts the plane $x=0$ at **T**, we find the coordinates of **T** by putting $x=0$ in the equation of the line.

Hence $$\frac{-x_1}{\cos\alpha}=\frac{y-y_1}{\cos\beta}$$

$$\therefore\ y=y_1-\frac{x_1\cos\beta}{\cos\alpha}=\mathrm{OV}$$

Similarly $$z=z_1-\frac{x_1\cos\gamma}{\cos\alpha}=\mathrm{VT}$$

N.B.—(i.) The equation $\frac{x-2}{2}=\frac{y-1}{3}=\frac{z-4}{1}$ is the equation of a line through (2, 1, 4), and its direction cosines will be proportional to 2, 3, 1. As on p. 165 it will be found that the actual direction cosines are

$$\frac{2}{\sqrt{14}},\ \frac{3}{\sqrt{14}},\ \frac{1}{\sqrt{14}}$$

(ii.) The equation $\frac{x-2}{2}=\frac{y-1}{3}=\frac{z-4}{1}$

may be written $\frac{x-2}{2}=\frac{y-1}{3}$ and $\frac{y-1}{3}=\frac{z-4}{1}$

or $3x-2y-4=0$ and $y-3z+11=0$

These are the equations of two planes and the line is the line of intersection of these planes.

Comparing $3x-2y-4=0$ with $x\cos\alpha+y\cos\beta+z\cos\gamma=p$ we see that $\cos\gamma=0$ or $\gamma=90°$, the plane is therefore parallel to **OZ**. Similarly $y-3z+11=0$ is parallel to **OX**.

By manipulating two planes, one parallel to one axis and the other parallel to a second axis, it will be seen that a line in any position may be obtained by their intersection.

Equation of a Straight Line joining Two Fixed Points.

If the two fixed points are **A** $(x_1y_1z_1)$, **B** $(x_2y_2z_2)$, it is clear from a figure that

$$\mathrm{AB}\cos\alpha=x_2-x_1;\ \mathrm{AB}\cos\beta=y_2-y_1;\ \mathrm{AB}\cos\gamma=z_2-z_1$$

Hence cos α, cos β, cos γ are proportional to x_2-x_1, y_2-y_1, z_2-z_1.

Also $\mathbf{AP}\cos\alpha=x-x_1$; $\mathbf{AP}\cos\beta=y-y_1$; $\mathbf{AP}\cos\gamma=z-z_1$.

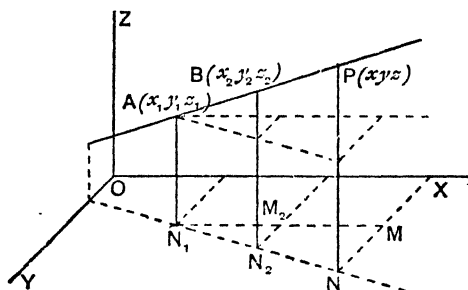


FIG. 133

$$\therefore \frac{\mathbf{AP}}{\mathbf{AB}}=\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Hence the required equation of the line is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

This result can, of course, be deduced at once from the previous paragraph.

The equation of a line through $x_1y_1z_1$ is

$$(\alpha)\quad \frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\cos\beta}=\frac{z-z_1}{\cos\gamma}$$

If the point $(x_2y_2z_2)$ is also on the line we have

$$(\beta)\quad \frac{x_2-x_1}{\cos\alpha}=\frac{y_2-y_1}{\cos\beta}=\frac{z_2-z_1}{\cos\gamma}$$

$\therefore$ cos α, cos β, cos γ are proportional to x_2-x_1, y_2-y_1, z_2-z_1.

From α and β we have

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

When the equation of a line is given as the intersection of two planes, say $2x-3y+8=0$, $4y-2z-18=0$, and it is

required to write it in the form $\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\cos\beta}=\frac{z-z_1}{\cos\gamma}$, we must find first the co-ordinates of some point on the line. Take any value for x, say $x=2$, then $y=4$ and $z=-1$.

Now write the given equations in terms of $x-2$, $y-4$, and $z+1$, thus:

$$2(x-2)-3(y-4)=0 \text{ and } 4(y-4)-2(z+1)=0$$

$$\therefore \frac{x-2}{3}=\frac{y-4}{2}=\frac{z+1}{4}$$

and the direction cosines are $\frac{3}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{4}{\sqrt{29}}$.

The result obtained will, of course, vary according to the point we select. If, for instance, we take $x=0$ we get $y=\frac{8}{3}$, $z=-\frac{11}{3}$, and the equation becomes

$$\frac{x}{3}=\frac{y-\frac{8}{3}}{2}=\frac{z+\frac{11}{3}}{4}$$

EXAMPLE 1. Find whether the line (i.) $\frac{x+1}{2}=\frac{y-3}{-1}=\frac{z-5}{-2}$ intersects the line (ii.) $\frac{x-4}{-3}=\frac{y-3}{-1}=\frac{z-4}{-1}$.

Let
$$\frac{x+1}{2}=\frac{y-3}{-1}=\frac{z-5}{-2}=k$$

Then $x=2k-1$; $y=-k+3$; $z=-2k+5$ represent the co-ordinates of points on (i.). If any such point is also on (ii.) we have

$$\frac{2k-5}{-3}=\frac{-k}{-1}=\frac{-2k+1}{-1}$$

and by solving it will be found that $k=1$.

Hence $x=1$, $y=2$, $z=3$, is on both lines.

Equation (i.) can be written $\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z-3}{-2}$ (by subtracting 1 from each fraction), and equation (ii.) is $\frac{x-1}{-3}=\frac{y-2}{-1}=\frac{z-3}{-1}$ (by subtracting 1 from each fraction).

EXAMPLE 2. Find the equation of the line which is the shortest distance between the lines

$$\text{(i.)}\ \frac{x-1}{3}=\frac{y-2}{4}=\frac{z-3}{5}\ \text{and}\ \text{(ii.)}\ \frac{x-2}{2}=\frac{y-3}{3}=\frac{z-1}{2}$$

The required line must be perpendicular to both the given lines, and as its equation involves the co-ordinates of some point on it we take the point (a, b, o) where the line cuts the plane $z=0$ and assume for its equation

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z}{n}$$

Since this is perpendicular to (i.) and to (ii.) we have

$3l+4m+5n=0$ and $2l+3m+2n=0$. (See p. 168.)

$$\therefore\ l:m:n=\ -7:4:1$$

$$\therefore\ \frac{x-a}{-7}=\frac{y-b}{4}=\frac{z}{1}\ \text{meets (i.) and (ii.)}$$

$$\therefore\ \frac{-7z+a-1}{3}=\frac{4z+b-2}{4}=\frac{z-3}{5}\quad \text{(iii.)}$$

and

$$\frac{-7z+a-2}{2}=\frac{4z+b-3}{3}=\frac{z-1}{2}\quad \text{(iv.)}$$

By eliminating z, we get from (iii.) $\frac{5a+4}{38}=\frac{5b+2}{-16}$,

and from (iv.) $\frac{a-1}{8}=\frac{-2b+3}{5}$.

$$\therefore\ a=-\frac{775}{33},\quad b=\frac{302}{33}$$

EXAMPLES XXIX

1. Find the equation of a straight line through (2, 3, 1) whose direction cosines are $\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}$.

2. Find a line through (1, 2, −1) which makes equal angles with the three axes.

3. Find the equation of a line through (2, 1, −1) whose direction cosines are proportional to 1, 3, 1.

4. Find the direction cosines of the line joining (7, 4, 3) and (3, 5, −2).

5. Find the equations of the lines joining the points (i.) (3, 1, 2), (−2, 3, 4); (ii.) (−2, 0, 1), (3, 2, −4). Find the co-ordinates of the point where (i.) meets the plane $z=0$ and where (ii.) meets the plane $x=0$.

6. Find the angle between the line joining (1, −1, 2), (−2, 3, 1) and the perpendicular to the plane $x-2y+2z=3$. What is the angle between the line and the plane?

7. Find the length of the projection of the line joining (−1, 2, 0) and (1, −1, 2) on the plane $2x-y-2z=4$.

8. Find the equation of the median through **A** of the triangle **A** (3, 4, 5), **B** (4, 5, 3), **C** (5, 3, 4).

9. **AB**=15 feet, **AC**=12 feet, are two sides of the floor of a room whose height **AD** is 12 feet. **E** is 3 feet from **DAB**, 2 feet from **DAC**, 3 feet from the floor. The corresponding distances for **F** are 4 feet, 5 feet, 5 feet. Find where the line joining **EF** will meet (i.) the floor; (ii.) the wall **DAB**; (iii.) the wall **DAC**; (iv.) the wall opposite **DAB**; (v.) the ceiling.

10. Find the equation of the line through **D** (1, 1, 5), perpendicular to the plane through **A** (0, 1, 1), **B** (4, 2, 2), **C** (−3, −3, 2), and find the co-ordinates of the point where the line meets the plane.

11. Show that the equations $\frac{x+2}{2}=\frac{y+8}{5}=\frac{z+5}{3}$ represent the same line as $\frac{x-2}{2}=\frac{y-2}{5}=\frac{z-1}{3}$.

12. Show that the lines $(8x-4y-4z=0,\ 7x+10y-8z=0)$ and $(3x+2y+z=5,\ x+y-2z=3)$ are perpendicular.

13. Borings are made at **O**, **A**, **B** on a horizontal plane to meet a geological stratum, which is reached at a depth of 50 feet at **O**, 80 feet at **A**, 130 feet at **B**. **A** is 300 feet due E. of **O**, **B** is 300 feet due N. of **O**. Considering the stratum to be a plane, find the depth at which it will be reached at **C**, 100 feet E. of **O** and 100 feet N. of **O**. Find also the slope of the stratum.

14. A diagonal is drawn on a rectangular sheet of paper **ABCD** when **AB**=12 inches, **BC**=8 inches. The paper is then folded along the perpendicular bisector of **BC** so that the two parts are at right angles to each other. Find the angle between the two parts of the diagonal.

15. The shaft of a coal mine is 250 yards deep, and a gallery 130 yards long runs from its lower end in a direction S, 40° W., sloping downwards at 12°. A second gallery leading from the end of the former is 280 yards long, has a direction, N, 25° W. and slopes downwards at 10°. Taking rectangular axes at the pit mouth in directions E, N, and vertically downwards, find the co-ordinates of P, the far end of the second gallery. If a straight line were drawn from the pit mouth to P, find its length, bearing and inclination to the vertical.

16. Taking as reference planes two vertical walls which meet at right angles, the co-ordinates of the tops of two vertical poles are (3·2, 4·5, 6) and (7·3, 7·0, 9). Find where a wire stretched between these two points would meet the ground.

17. A cord is stretched between two points in a room. If the three planes of reference are (i.) the N. wall; (ii.) the E. wall; (iii.) the floor, the co-ordinates of the two points are (5, 2, 4½) and (4, 5½, 1½). The unit being 1 foot, find where the cord, if prolonged, would intersect the floor and the east wall.

18. Two walls of a room meet the floor in OX and OY, and meet one another in OZ. ABCD is a blackboard placed with AB on the floor parallel to OX and 4 feet from it, so that AD touches the wall ZOY and DC leans against the wall ZOX. If AB=3 feet, AD=5 feet, find (i.) the equation of the plane ABCD; (ii.) the direction cosines and equations of the diagonals BD and AC.

19. Find whether the line $\frac{x}{1}=\frac{y+3}{2}=\frac{z-4}{-2}$ meets the line $\frac{x+1}{-2}=\frac{y+2}{-1}=\frac{z-5}{3}$, and if so, write both equations to show that the lines represented pass through the point of intersection.

20. Find the equation of the line which is the shortest distance between the lines $4x=3y=-z$ and $3(x-1)=-y-2=-4z+2$.

Equation of a Plane passing through the Line of Intersection of two Given Planes.

Let the equations of the given planes be $Ax+By+Cz=1$ and $A^1x+B^1y+C^1z=1$. Suppose XYZ to be any point on the line of intersection, this point will be on both planes, therefore (i.) $AX+BY+CZ=1$ and (ii.) $A^1X+B^1Y+C^1Z=1$.

Now consider the equation

$$Ax+By+Cz-1+k(A^1x+B^1y+C^1z-1)=0$$

Since it is of the first degree it represents a plane, also $AX+BY+CZ-1+k(A^1X+B^1Y+C^1Z-1)$ will be zero from (i.) and (ii.).

$\therefore$ XYZ is on this plane.

It must therefore be a plane containing the line of intersection of the given planes.

Length of the Perpendicular from a Point to a Plane.

Let $x \cos \alpha + y \cos \beta + z \cos \gamma = p$ be the equation of the plane and $x_1y_1z_1$ the co-ordinates of P. Suppose a plane drawn through P parallel to the given plane. The perpendiculars from the origin will be along the same line, and if p^1 is the length of the perpendicular from P to the given plane the perpendicular from the origin to the parallel plane will be $p+p^1$, and its equation will be

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p + p^1$$

But $x_1y_1z_1$ is on this plane.

$$\therefore x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma = p + p^1$$

$$\therefore p^1 = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p$$

Compare p. 51.

If the given plane is $Ax+By+Cz=1$ it must be written in the perpendicular form and the value of p^1 will be

$$\frac{Ax_1+By_1+Cz_1-1}{\sqrt{A^2+B^2+C^2}}$$

Length of the Perpendicular from a Point to a Line.

The general case leads to a complicated formula and the method will be shown by taking an example.

Suppose the equation of the line to be

$$\frac{x-2}{2}=\frac{y-3}{5}=\frac{z-1}{3}$$

and that the perpendicular is to be drawn from **P** (3, 4, 5). It is clear that **A** (2, 3, 1) is a point on the line.

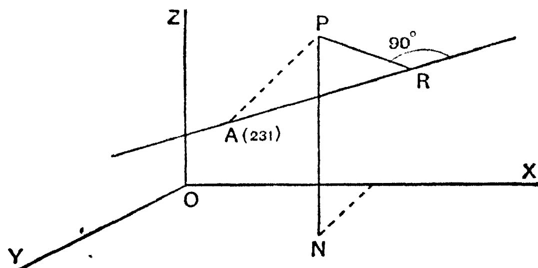


FIG. 134

The length of AP^2 is $(3-2)^2+(4-3)^2+(5-1)^2$.

If **PR** is perpendicular to the line and $PAR=\theta$, then $PR=AP\sin\theta$.

The direction cosines of **AP** are proportional to $3-2$, $4-3$, $5-1$—*i.e.* 1, 1, 4—and are therefore $\frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}}$.

The direction cosines of **AR** are $\frac{2}{\sqrt{38}}, \frac{5}{\sqrt{38}}, \frac{3}{\sqrt{38}}$.

$$\therefore \cos\theta=\frac{1}{\sqrt{18}}\cdot\frac{2}{\sqrt{38}}+\frac{1}{\sqrt{18}}\cdot\frac{5}{\sqrt{38}}+\frac{4}{\sqrt{18}}\cdot\frac{3}{\sqrt{38}}$$

$$=\frac{19}{\sqrt{18\times 38}} \qquad \therefore \theta=43^\circ\ 24'$$

Hence $$PR=2{\cdot}9$$

EXAMPLE 1. Find the equation of a plane containing **OZ** and the point (2, 3, 1).

The axis **OZ** is the intersection of the plane $x=0$ and the plane $y=0$. Any plane through their line of intersection will have an equation of the form $x+ky=0$. Since this is to contain (2, 3, 1) we have $2+3k=0$.

$\therefore\ k=-\frac{2}{3}$, and the required equation is $3x-2y=0$

EXAMPLE 2. Find the length of the perpendicular from (1, 2, 3) to the plane $4x-3y+z=2$.

Writing the equation in the form

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p$$

we have
$$\frac{4x}{\sqrt{26}}-\frac{3y}{\sqrt{26}}+\frac{z}{\sqrt{26}}=\frac{2}{\sqrt{26}}$$

The perpendicular from (1, 2, 3) is therefore

$$\frac{4(1)-3(2)+3-2}{\sqrt{26}}=-\frac{1}{\sqrt{26}}$$

and is on the same side of the plane as the origin, since the perpendicular from the origin is also negative $\left(=-\frac{2}{\sqrt{26}}\right)$.

EXAMPLES XXX

1. Find the length of the perpendicular from (1, 3, 4) on the plane which makes intercepts of 4, 5, 3 on the axes.

2. A plane passes through the points **A** (0, 1, 3), **B** (1, 0, −4), **C** (1, 1, −1). Find its equation and the length of the perpendicular to it from (2, 4, −3).

3. Find the equation of the plane passing through the intersection of $3x+2y-z=4$ and $x-3y+2z=1$, and also through the point (3, 4, 5).

4. A plane passes through the point (2, 1, 4) and is at right angles to the line of intersection of $5x+y+3z=0$ and $3x-y+z+1=0$. Prove that the point (2, 3, 5) lies on it.

5. Find the point of intersection of the planes $4x-3y+2z=4$, $x+y+3z=1$, $2x-y-z=2$.

6. Find the equation of the plane which contains the **X** axis and the point (3, 2, 5).

7. Three edges of a rectangular box are **OA** (6 inches), **OB** (5 inches), **OC** (4 inches). If **P** is the opposite angular point to **O**, find the distance of **P** from the plane through **A**, **B** and **C**. If this perpendicular were produced, find how far from **OA** and **OB** it would meet the plane **OAB**.

8. Find the length of the perpendicular from (1, 1, 2) to the plane $2x-3y+z=5$.

9. Find the equation of the plane through (2, 3, -1) parallel to $3x-4y+7z=0$.

10. Find the equation of the plane through the origin and containing the line of intersection of the two planes $5x-3y+2z+5=0$ and $3x-5y-2z=7$.

11. Three edges of a rectangular box, **OX**, **OY**, **OZ**, are taken as axes. If **OX**$=6$ inches, **OY**$=4$ inches, **OZ**$=3$ inches, find the equation of the plane containing the vertical edge through **X** and the mid-point of the edge parallel to **OX** on the other side of the box.

12. Find the equation of the line in which the plane found in in Qu. 11 meets the plane parallel to **OX** which contains the diagonal **ZY** of one of the faces.

13. Find the equation of the planes parallel to $3x-y+2z=12$ which are 2 units away from it.

14. Find the equation of the plane through (3, 1, 2) which is parallel to the plane $3x-2y-6z=6$.

15. The angular points of a tetrahedron are **A** (3, 4, 2), **B** (1, 2, 1), **C** (4, 1, 3), **D** (-1, -1, 3). Find the distance of **D** from the face **ABC**.

If **P** (3, 2, 3) and **Q** (7, 4, 5) are two points in space, and pq, p^1q^1 the projections of **PQ** on the planes $z=0$ and $y=0$,

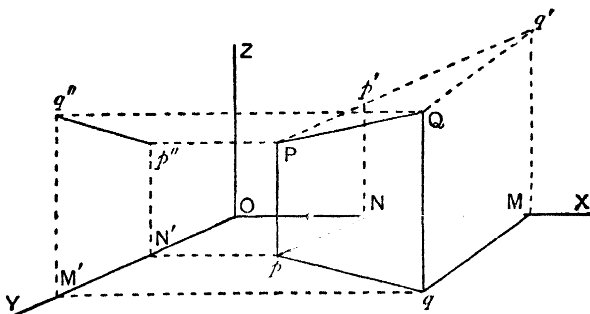


Fig. 135

the correct lengths of these projections may be drawn by supposing the plane **YOX** to be rotated round **OX** until it

is in the same plane as **ZOX**; p^1**N** and p**N**, also q^1**M** and q**M** will then come in the same straight line.

If **ZOY** be supposed to be rotated in a similar way

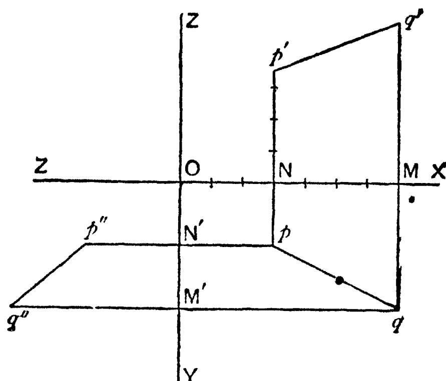


FIG. 136

round **OY** the projections $p''q''$ on **ZOY** may also be drawn.

The point where a straight line meets one of the co-

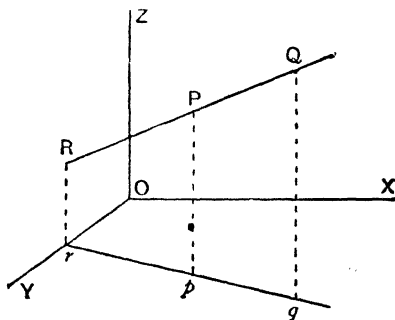


FIG. 137

ordinate planes (**ZOY**, say) may be found graphically by noting that when the line **QP** meets **ZOY** at **R**, its projection

qp meets OY at r, and its projection q^1p^1 meets OZ at r^1.

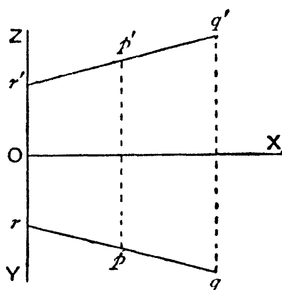


FIG. 138

The co-ordinates of the point R are therefore $x=0$, $y=\text{O}r$, $z=\text{O}r^1$. (Fig. 138.)

Similarly for the other planes.

Equations of the Second Degree.

We have seen that the equation of a straight line in space is given by the equations of two planes. The co-ordinates of every point on the line satisfy both equations, and every point whose co-ordinates satisfy both equations lies on the line.

Two surfaces in general intersect in a curve and the equations of the two surfaces will form the equation of the curve of intersection.

For instance, in two dimensions the equation of a circle is $x^2+y^2=a^2$, but in three dimensions this will be the equation of a cylinder, since for any value of z, the points whose distance from the Z axis is a will be such that $x^2+y^2=a^2$. The equations of the circle in which this cylinder cuts the plane XOY will be $x^2+y^2=a^2$; $z=0$.

EXAMPLE 1. Find the equations of the locus of a point whose distance from the origin is 3 and which is equidistant from the points 4, 0, 0 and 0, 4, 0.

If **P** (xyz) be any point on the locus the first condition gives $x^2+y^2+z^2=9$, which is evidently the equation of a sphere.

The second condition gives

$$\sqrt{(x-4)^2+y^2+z^2}=\sqrt{x^2+(y-4)^2+z^2}$$
$$\therefore\ x-y=0$$

This is the equation of a plane which contains the **Z** axis and which intersects **XOY** in the line $y=x$. The required locus is the intersection of this plane and the sphere, and is therefore a circle.

EXAMPLE 2. Find the curve determined by the intersection of the plane $z=2$ and the surface $y^2+z^2=4x$.

Since the co-ordinates of points on the curve of intersection satisfy both these equations we may write $z=2$ in $y^2+z^2=4x$ *i.e.* $y^2=4x-4$.

Now $y^2=4x-4$ is the equation of a parabola in two dimensions, hence in three dimensions it represents a cylinder with a parabolic surface which cuts the plane **XOY** in the curve $y^2=4x-4$. The curve given by $z=2$ and the surface $y^2+z^2=4x$ is therefore the parabola in which the plane $z=2$ intersects this cylinder.

EXAMPLE 3. Find the nature of the surface given by the equation $y^2+z^2=4x$.

The surface passes through the origin since the equation is satisfied by (0, 0, 0) and it does not cut the axes again.

The equation being unaltered when $-z$ is written for $+z$ the curve is symmetrical about the plane **XOY**. Similarly it is symmetrical about **XOZ**, and therefore about **OX**.

It cuts (i.) the plane **XOY** $(z=0)$ in the parabola $y^2=4x$; (ii.) the plane **XOZ** in the parabola $z^2=4x$; (iii.) the plane **YOZ** in the curve $y^2+z^2=0$, which represents a point circle (radius being zero).

The surface is cut by the plane $x=a$ (parallel to **ZOY**) in the circle $x^2+z^2=4a$; the greater a is taken, the greater becomes the radius of this circle.

The surface is cut by the planes $y=b$ and $z=b_1$, in the parabolas $z^2=4x-b^2$ and $y^2=4x-b_1^2$.

Paraboloid of Revolution.

When a parabola is rotated about its axis the surface generated is called a paraboloid of revolution.

The equation of the parabola **POQ** lying in the plane $z=0$ is $y^2=4ax$—*i.e.* **PM**$^2=4a$ **MO**.

Suppose this parabola rotated about **OX** so that **MP** moves upwards through the angle **NMP**. The relation

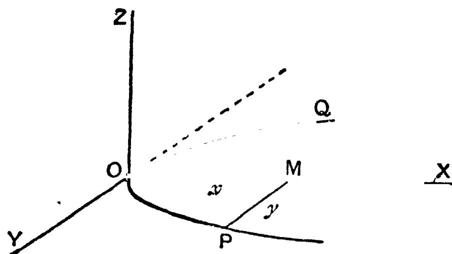


FIG. 139

PM$^2=4a$. **OM** still holds good, but if x, y, z are the co-ordinates of **P** we have **PM**$^2=y^2+z^2$.

Hence the equation of the paraboloid is $y^2+z^2=4ax$.

If the parabola is drawn with **OY** as its axis its equation is $x^2=4ay$, and when it is rotated about **OY** the equation

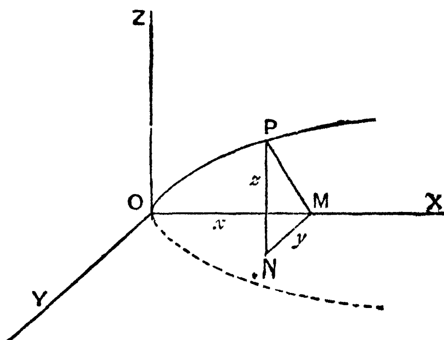


FIG. 140

of the resulting paraboloid will be $x^2+z^2=4ay$. Similarly a paraboloid with **OZ** as its axis will have the equation $x^2+y^2=4az$.

Ellipsoid of Revolution.

If an ellipse whose centre is at the origin has its major axis along OX and lies in the plane $z=0$, its equation will be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$$

If this is rotated about its major axis it will be seen that the distance of any point P from the major axis will become $\sqrt{y^2+z^2}$ instead of y, and the equation of the ellipsoid of revolution will be $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$. This is known as a prolate spheroid.

If the curve is rotated about its minor axis which lies along OY the equation of the solid generated will be $\frac{x^2+z^2}{a^2}+\frac{y^2}{b^2}=1$. This is called an oblate spheroid.

If in the ellipsoid of revolution given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}=1$ we shorten each z in the ratio $\frac{c}{b}$ we obtain a solid known as the ellipsoid whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

The section of this solid made by each of the three co-ordinate planes will be an ellipse.

Hyperboloid of Revolution.

The surface obtained by revolving the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $z=0$ about its minor axis has the equation

$$\frac{x^2+z^2}{a^2}-\frac{y^2}{b^2}=1$$

This surface will be all in one piece and is called a hyperboloid of revolution of one sheet.

If the curve is rotated about its major axis it will be

in two pieces and its equation is $\frac{x^2}{a^2}-\frac{y^2+z^2}{b^2}=1$. It is called a hyperboloid of revolution of two sheets.

If the z of the solid $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{a^2}=1$ is reduced in the ratio $\frac{c}{a}$

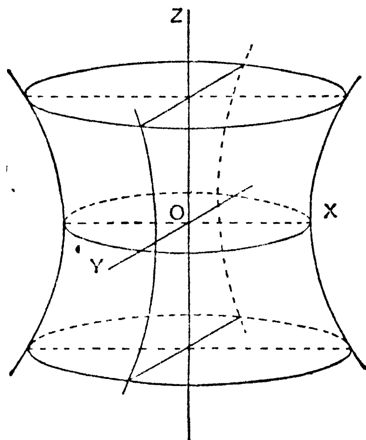


FIG. 141

the solid is a hyperboloid of one sheet whose equation is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Fig 141 shows the solid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

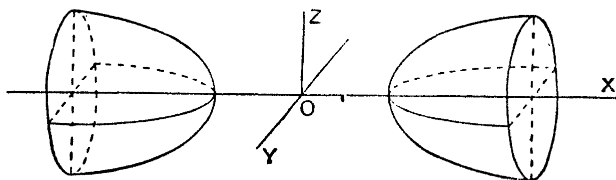


FIG. 142

Similarly the equation of the hyperboloid of two sheets is $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. (Fig. 142.)

Cone of Revolution.

If the line $y=mx$, $z=0$ is rotated about the **X** axis we obtain the cone whose equation will be $\sqrt{y^2+z^2}=mx$ or

$$y^2+z^2=m^2x^2$$

EXAMPLES XXXI

1. Find the equation of the locus of a point equidistant from (6, 4, 3) and (6, 4, 9) and also from (−5, 8, 3) and (−5, 0, 3).

2. Find the equation of the locus of a point 2 units above the **XY** plane and 3 units from (2, 1, 4).

3. Find the equation of the locus of a point 4 units from the origin and equidistant from (3, 1, 2), (−3, −1, −2).

4. Find the equation of a sphere whose centre is at (2, 0, −1) and which passes through (1, 4, −2).

5. State the equations of the curves in which the following surfaces cut the co-ordinate planes:—(i.) $x^2+4y^2+9z^2=16$; (ii.) $x^2+4y^2-16z^2=0$.

6. Find the equation of the solid generated when the line $y=mx+b$, $z=0$ revolves about (i.) the **X** axis; (ii.) the **Y** axis.

7. Prove that the equation $Ax^2+By^2+Cz^2=0$ is the equation of a cone by showing that if a point $P(x_1y_1z_1)$ is on it then any point on the line joining **P** to the origin is also on it.

8. Find the equations of the curves in which the surface $\frac{x^2}{a}+\frac{y^2}{b}=4z$ is cut by planes parallel respectively to the co-ordinate planes. The solid is known as an elliptic paraboloid. Give corresponding results for the hyperbolic paraboloid

$$\frac{x^2}{a}-\frac{y^2}{b}=4z$$

9. What solids are represented by the following equations: —(i.) $x^2+4y^2+9z^2=36$; (ii.) $x^2+z^2=16$; (iii.) $x^2+z^2=4x$.

10. If r_1 be the semi-diameter of an ellipsoid whose direction cosines are l_1 m_1 n_1, prove that $\frac{1}{r_1^2}=\frac{l_1^2}{a^2}+\frac{m_1^2}{b^2}+\frac{n_1^2}{c^2}$. If r_1 r_2 r_3 are three semi-diameters mutually at right angles prove $\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

MISCELLANEOUS EXAMPLES

R

1. A right triangular prism rests with its base **ABC** on a horizontal plane. Points **D**, **E**, **F** are taken on the vertical edges above **A**, **B**, and **C**, so that **AD**=10 inches, **BE**=8 inches, **CF**=18 inches. Find the height of the centroid of the section **DEF** above the horizontal plane **ABC**.

2. **OX**, **OY**, **OZ** are three edges of a rectangular box whose base is **OXAY** and whose top is **ZBCD**. If the co-ordinates of **C** referred to **OX**, **OY**, **OZ** as axes are 4, 5, 3, find the equations of **CX** and **CY** and the angle **XCY**.

3. Find the equation of the plane bisecting at right angles the line joining 3, 4, 6 to the origin.

4. Find the equation of the locus of a point which is equidistant from the points **A** (1, 2, 4) and **B** (4, 3, 2).

5. The plane $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ cuts the axes in the points **A**,**B**,**C**. Find the area of **ABC**.

S

1. The co-ordinates of 3 points being **A** (2, 4, 5), **B** (1, 2, 3), **C** (3, 6, 2), find the length of the projection of **AB** upon **OC**.

2. Find the equations of the line joining **A** (1, −2, 3) and **B** (−2, 3, 1). Find the angle which its projection on the **XY** plane makes with the **X** axis.

3. From the point **A** (p, q, r) perpendiculars **AP** and **AQ** are drawn to the planes **YOZ** and **XOZ** respectively. Find the equation of the plane parallel to **OZ** which passes through **P** and **Q**.

4. A man stands on a low roof with his eye 17 feet above the ground opposite a rectangular wall **ABCD**, of which **AB** and **CD** are the vertical ends. The angles of elevation of the points **A** and **C** are 45° and 20° respectively, and the plane containing his eye **E** and **AB** makes an angle of 45° with a vertical plane perpendicular to the wall. His distance from the wall is 70 feet. Take as axes **OX**, **OY**, **OZ**, the lines through his eyes (i.) perpendicular to the wall; (ii.) horizontal and parallel to it; (iii.) vertical. Find the co-ordinates of **A**, **B**, **C**, **D**; the equation of **AC**; the area of **ABCD**.

5. A sphere is described with its centre at the origin and radius 7. Find the equation of the plane which touches the sphere at the point (2, 3, 6).

T

1. A rectangular board **ABCD** is supported so that the co-ordinates of three of its corners are **A** (2, 3, 4), **B** (5, 4, 6), **C** (8, 7, 9). Find the co-ordinates of the mid-point of the diagonal **AC** and hence find the co-ordinates of the other corner **D**.

2. Find the direction cosines of the line in which the plane of the blackboard in Qn. 1 is cut by a plane which contains the **Z** axis and the point **C**.

3. A straight line joins the points **P** (—2, —1, 3) and **Q** (3, 2, —1). Find the angle it makes with **OX** and the length of its projection on **OX**.

4. **OX**, **OY** are two edges at right angles of the bottom of a sloping desk whose lid is a plane passing through a line parallel to **OY** and 4 inches above it. The gradient of the lid is 1 in 5 and the line of its hinges is 8 inches above the plane **XOY**. Find the equation of the lid and find whether a point 6 inches from **ZOY** and $5\frac{1}{2}$ inches above **XOY** will be above or below the lid. Find the length of the perpendicular from this point to the lid.

5. The lines $\frac{x}{2}=\frac{y+1}{2}=\frac{z}{1}$ and $\frac{x-2}{2}=\frac{y-1}{3}=\frac{z-1}{4}$ intersect. Find the distance of the point 4, 3, 2 on the first line from the point of intersection of the lines.

U

1. Find the equation of the plane parallel to the line $\frac{x-2}{1}=\frac{y-1}{3}=\frac{z-3}{2}$, which contains the points (—3, 1, 2) (0, 0, 0),

2. Find the equation of the line **AB** in which the plane $2x+y+3z=17$ meets the plane **XOY**, and find the **Z** co-ordinate of the point **P** on the former plane for which $x=1$ and $y=3$. Let the point on the plane **XOY** vertically under **P** be **N**. Find the length of the perpendicular **NK** drawn from **N** to the line **AB** and find also the co-ordinates of **K** and the inclination of the line **PK** to the plane **XOY**.

3. A plane makes intercepts of 1, 3, 2 on the axes **OX**, **OY**, **OZ**. Another plane perpendicular to it contains the axis **OZ**. Find the direction cosines of the line of intersection of the two planes.

4. Take a point **A** in the axis **OX**, draw **AB** in the plane **ZOX**, making 60° with the positive direction of **OX** and **AC** in the plane **XOY**, making also 60° with **OX**. If **AB** is 4 feet long and represents the elevation of a stick **AD**, of which **AC** is the plan, find the direction cosines of the stick, and its length.

5. Referred to the intersection of two walls perpendicular to one another with the ground, the co-ordinates of the tops of 2 posts fixed in the ground are **A** (5, 5, 7), **B** (8, 4, 9). A wall making 45° with the positive direction of **OX** meets **OX** where $x=6$ and a rope stretched from **A** to **B** just reaches the top of this wall Find its height.

A roof from the edge of the wall rests on the tops of the two poles and is brought down to the other two walls. Find equations of its lines of intersection with the walls.

ANSWERS

CHAPTER I

Examples I. (p. 4)

1. 5, 3, 8. **2.** $\sqrt{5}$; 5; $\sqrt{13}$. **3.** (2, 4) (1, 4).
4. Parallel lines. **5.** AC = BC = $\sqrt{65}$.
6. AB = DC = $\sqrt{40}$; AD = BC = $\sqrt{72}$.
7. $\sqrt{x^2+(y-1)^2}$; $x^2+y^2-2y=15$.
8. $\sqrt{(x+2)^2+y^2}$; $\sqrt{(x-2)^2+y^2}$; $x=0$. P lies on OY.
9. $\sqrt{13}$. **10.** $3\sqrt{2}$; 3; 3. **11.** $\left(\frac{8}{3}, \frac{8}{3}\right)$.
12. $\sqrt{(x-2)^2+y^2}$.

Examples II. (p. 10)

1. $x=\pm 2$. **2.** $x^2+y^2=16$. **3.** $y=2x$.
4. $y=\pm\sqrt{3}x$. **5.** $x^2-3y^2-2x-4y+5=0$.
6. $10x+4y-27=0$. **7.** $3x^2+3y^2-16x-8y+20=0$.
8. $x^2-2x+y^2-4y-4=0$. **9.** $y^2-4y-6x+1=0$.
10. $x^2-4x+4y-8=0$. **11.** $x+y=2$.
12. $3x^2+3y^2-10x+3=0$. **13.** $x^2+y^2=25$.
14. $x^2+y^2=16$. **15.** $3x-2y+6=0$. **16.** $y=3x$.
17. $y=2x+1$. **18.** $y=4x-8$.

Examples III. (p. 13)

1. (i.) $2x-y-1=0$; (ii.) $2x-y+7=0$; (iii.) $x-y-1=0$;
(iv.) $x-y-5=0$; (v.) $x-2y=0$; (vi.) $2x-3y+6=0$.
2. (i.) $3x-2y=0$; (ii.) $x-2y+5=0$; (iii.) $2x-3y+7=0$;
(iv.) $x-y-1=0$; (v.) $2x-y-5=0$; (vi.) $5x-6y-2=0$.
3. (i.) $\left(\frac{2}{3}, \frac{8}{3}\right)$; (ii.) (−·8, 1·8) (1·8, −·8); (iii.) (0, 0) (1, 2).
4. $\left(-\frac{1}{5}, -\frac{2}{5}\right)$. **5.** $3x-y-5=0$. **6.** $y=\pm\frac{3}{4}x$.
7. (2, 0) (3, −2). **8.** (4, 2). **9.** No.
10. $3x-2y+1=0$. **11.** 0; −2; (0, −4).
12. (0, −1) (1, 0). **13.** $\sqrt{40}$. **14.** $xy=10$. **15.** $\sqrt{13}$.

CHAPTER II

Examples IV. (p. 23)

1. (i.) $-2, 3$; (ii.) $\frac{1}{2}, 3$; (iii.) $-\frac{1}{2}, 2$; (iv.) $\frac{3}{2}, -3$.

2. (i.) $y=2x+1$; (ii.) $y=-2x+1$; (iii.) $y=-2x-1$; (iv.) $x-2y+3=0$.

3. (i.) $3x+2y-6=0, -\frac{3}{2}$; (ii.) $3x-2y+6=0, \frac{3}{2}$; (iii.) $3x+2y+6=0, -\frac{3}{2}$.

4. (i.) $y=\frac{x}{\sqrt{3}}+2$; (ii.) $y=\sqrt{3}x+1$; (iii.) $y=-\sqrt{3}x+1$. (iv.) $y=-\sqrt{3}x-1$.

5. (i.) $2x-y-1=0$; (ii.) $2x-y+7=0$; (iii.) $2x-y+1=0$; (iv.) $2x+y+7=0$; (v.) $2x+y-1=0$; (vi.) $x-2y=0$; (vii.) $x+2y=0$.

6. (i.) $3x-2y=0$; (ii.) $x-2y+5=0$; (iii.) $2x+3y+8=0$ (iv.) $x+y-2=0$; (v.) $2x-6y+1=0$.

7. $\frac{x}{a}+\frac{y}{b}=1$. **8.** $4, 3$; $\frac{12}{5}$.

9. 2, 1. $\tan\alpha=\frac{1}{3}$; $\alpha=18°\ 26'$.

10. (i.) $(3, 5)$; (ii.) $\left(\frac{10}{3}, \frac{17}{3}\right)$; (iii.) $\left(\frac{8}{3}, \frac{13}{3}\right)$.

(i.) $(1, -2)$; (ii.) $\left(2, -\frac{11}{3}\right)$; (iii.) $\left(\frac{5}{2}, -\frac{9}{2}\right)$.

11. (i.) $(6, 11)$; (ii.) $(5, 9)$.

12. (i.) $y=x-2$; (ii.) $x-2y-1=0$; (iii.) $2x+y+4=0$.

13. **AB**, $3x-5y-1=0$; **BC**, $y+2=0$; **CA**, $3x+y-7=0$.

14. $3x-2y-4=0$; $3x-11y-13=0$; $3x+7y+5=0$, $\left(\frac{2}{3}, -1\right)$.

15. $2x+y-5=0$. **16.** $x+y=4$. **17.** $x=0$; $4x-3y=0$.

18. $c=5+\frac{3n}{2}$. A straight line. $\frac{3}{2}$.

19. $a=0{\cdot}2$, $b=15$. $P=115$.

20. $s=vt$. A straight line through the origin, gradient v. $s=vt+b$. A straight line, gradient v, intercept on **OY** $=b$.

21. 4·07 ft., 13·82 ft. **22.** 20 ft. behind **OP**, 4·2 ft. behind **OR.**

23. $\frac{5}{2}$ mls. E., $7\frac{1}{2}$ mls. N. **24.** $1\frac{3}{4}$. **25.** $\frac{10}{11}$. **26.** ·59, −3·39.

27. (8, 0). **28.** 6. **29.** $\sqrt{50}$.

30. $76\frac{2}{3}$ yds.; 300 yds. from **D**. **31.** $6\frac{2}{3}$ ft. 10 ft.

32. $\sqrt{160}$. **33.** $\frac{8}{3}, \frac{10}{3}$.

Examples V. (p. 28)

1. (i.) Positive $x>1$ or <-3, negative $1>x>-3$.
(ii.) Positive $x>-3$, negative $x<-3$.
(iii.) Positive $x>3$ or $x<-3$, negative $3>x>-3$.

2. $\frac{3}{2}>x>-2$.

3. (i.) Axes are asymptotes; (ii.) y negative when x is negative; (vi.) $x=5$ and $y=2$ are asymptotes; (vii.) $x=0$ and $y=1$ are asymptotes. Curve touches $y=0$ where $x=1$; (viii.) $y=3$ and $x=1$ are asymptotes; (ix.) $x \not> 2$; $x=0$ an asymptote. A cusp at $(2, 0)$; (x.) $x=-2$ and $y=\frac{3}{2}$ are asymptotes.

CHAPTER III

Examples VI. (p. 37)

1. (i.) $6x-y-3=0$; (ii.) $2x-y+1=0$; (iii.) $2x+y-3=0$; (iv.) $2x+y-5=0$; (v.) $y=1$; (vi.) $5x+12y-27=0$; (vii.) $y-y_1=-\frac{x_1+y_1}{x_1}(x-x_1)$; (viii) $y-y_1=\frac{x_1^3}{4}(x-x_1)$.

2. $x-yt+at^2=0$.

3. $y-y_1=-\frac{y_1}{x_1}(x-x_1)$; $\triangle=2c$.

4. $63° 26'$; $2{\cdot}1$.

5. $(-4, 0)$ $SP=ST=5$.

6. $yy_1=3(x+x_1)$; $(24, 12)$.

7. $\frac{5}{3}$; $y-\frac{3}{2}=\frac{5}{3}\left(x-\frac{5}{2}\right)$, $\tan\theta=\frac{x_1}{y_1}$; $Q\left(\frac{4}{x_1-y_1}, \frac{4}{x_1-y_1}\right)$; $R\left(\frac{4}{x_1+y_1}, \frac{-4}{x_1+y_1}\right)$; $\triangle=4$.

8. $x-4y+28=0$; $(28, 14)$.

9. $\left(\frac{9}{x_1}, 0\right)\left(0, \frac{9}{y_1}\right)$.

10. $\left(\frac{1}{2}, -2\right)$.

12. $3x-y-2=0$; $\left(\frac{2}{3}, 0\right)$ $(0, -2)$.

13. $-\frac{5}{8}$; $\left(\frac{-16a}{3}, \frac{+16a}{3}\right)$.

14. $y-32=144(x-3)$; $(0, 5)$ $(-1, 0)$ $(2, -27)$.

Examples VII. (p. 42)

1. (i.) $\frac{9}{2}$; (ii.) 13; (iii.) ∞.

2. (i.) $135°$, $18° 26'$, $26° 34'$; (ii.) $90°$, $33° 41'$, $56° 19'$.

3. $x+y=6$; $(0, 6)$.

4. $3x+2y-13=0$.

5. $2x+3y-17=0$.

6. $6x+2y-17=0$.

7. $y-y_1=\frac{2a}{y_1}(x-x_1)$; $y=-\frac{y_1}{2a}(x-a)$.

O

9. (i.) $\frac{4}{5}$; (ii.) 2; (iii.) $\frac{103}{17}$; (iv.) $\frac{6}{\sqrt{13}}$.

10. (i.) $ax-by=0$; (ii.) $x+my=0$; (iii.) $y=x\tan a$.

11. Sides $=\sqrt{10}$; $2x-y-4=0$; $x+2y-7=0$.

12. $x-x_1=\frac{a-x_1}{-y_1}(y-y_1)$; $x-x_1=\frac{a+x_1}{y_1}(y-y_1)$. Perpendicular if $\left(\frac{-y_1}{a-x_1}\right)\left(\frac{y_1}{a+x_1}\right)=-1$.

13. Take **C** (0, 0), **A** $(-a, 0)$, **B** (0, b).

14. $y-b=mx$; $y+b=-\frac{x}{m}$. **15.** 74° 45′; 34° 42′; 70° 34′.

16. $3x-y+11=0$; $x-y-5=0$; $3x-2y-2=0$; $(-8, -13)$.

17. $\left(\frac{16}{7}, \frac{15}{7}\right)$. **18.** $x+2y=7$; $x-2y+5=0$; 53° 7′.

19. $3x-11y+13=0$; $11x+3y-39=0$. **20.** $\frac{9}{\sqrt{10}}, \frac{21}{\sqrt{29}}, \frac{21}{\sqrt{29}}$.

21. (9, 12) (1, -4); $2x-3y+18=0$, $2x+y+2=0$; 29° 44′, 53° 8′.

22. $\left(\frac{3}{2}, \pm 2\right)$; $2x\pm y-5=0$; $2x\mp 4y+5=0$.

23. $x-2y+8=0$; 2, $-\frac{1}{2}$.

24. $5x-11{\cdot}8y+34{\cdot}75=0$; $5x+6{\cdot}8y+11{\cdot}5=0$; -4, 1·25.

CHAPTER IV

Examples VIII. (p. 49)

1. $3x-2y=6$. **2.** $x+y=6$; $y=x+2$.

3. (i.) $\frac{-3}{\sqrt{10}}x+\frac{y}{\sqrt{10}}=\frac{2}{\sqrt{10}}$; (ii.) $\frac{3x}{\sqrt{13}}+\frac{2y}{\sqrt{13}}=\frac{6}{\sqrt{13}}$; (iii.) $-\frac{4}{5}x+\frac{3}{5}y=\frac{1}{5}$.

5. $\frac{x}{2}+\frac{y}{2}=1$. **6.** (i.) 2; (ii.) 2. **7.** $y=\frac{x}{\sqrt{3}}$.

8. $y-5=\sqrt{3}(x-4)$; (2·33, 2·11).

9. $x\cos 60+y\sin 60=\frac{4+3\sqrt{3}}{2}$; $\frac{3\sqrt{3}}{2}$.

10. $5x-8y+15=0$ **11.** $\frac{x}{4}+\frac{y}{6}=1$.

13. $\frac{x}{OA}+\frac{y}{OB}=1$. **14.** $\frac{4}{5}x+\frac{3}{5}y=\frac{12}{5}$; $\frac{12}{5}$.

15. 233° 8′. **16.** 16° 16′.

Examples IX. (p. 55)

1. (i.) $\frac{3}{\sqrt{5}}$; (ii.) 0; (iii.) $\frac{13}{5}$; (iv.) $\frac{4x_1+3y_1+1}{5}$;

(v.) $\frac{2}{\sqrt{21}}(2\cos 30+3\sin 30-4)$; (vi.) $2\sin 60+\cos 60-2$.

2. (i.) 1; (ii.) $\frac{163}{13}$; (iii.) $\frac{a\sqrt{1+m^2}}{m}$. **3.** $\frac{7}{\sqrt{5}}$.

4. $\frac{11}{\sqrt{5}}, \frac{11}{\sqrt{34}}, \frac{11}{5}$. **5.** $2\sqrt{2}$. **6.** $\frac{6}{\sqrt{5}}$. **7.** $\frac{10}{3\sqrt{5}}$.

8. $\frac{3\mathbf{X}-4\mathbf{Y}-5}{5}$; $16\mathbf{X}^2+9\mathbf{Y}^2+24\mathbf{XY}-170\mathbf{X}-40\mathbf{Y}+375=0$.

9. (i.) Opposite; (ii.) opposite; (iii.) same; (iv.) opposite.

10. $-\frac{6}{\sqrt{5}}, +\frac{6}{\sqrt{5}}$. **11.** $\frac{12}{5}$. **12.** $\mathrm{PM}+\mathrm{PN}=\frac{2ab}{\sqrt{a^2+b^2}}$.

13. $\frac{11}{2}$. **14.** $5x+12y=39$; $5x+12y=-13$.

15. 2·1. **16.** $(-x, 0)$; $\sqrt{a(a+x_1)}$.

17. $\frac{bx_1-a^2}{\sqrt{x_1^2+y_1^2}}$; $\frac{-bx_1-a^2}{\sqrt{x_1^2+y_1^2}}$. **18.** $x_1^2+y_1^2=a^2$.

19. $3x+4y=25$. **20.** $p=\frac{ab}{\sqrt{a^2+b^2}}$ and $ab=a+b$.

Examples X. (p. 59)

1. (i.) $7x+y=0$, $x-7y=0$; (ii.) $x+y+8=0$, $x-y=0$.
2. $2x+y-3=0$, $x+3y-4=0$.
3. $x-2y+4=0$, $2x+y-12=0$.
4. (i.) $x+y-3=0$, $7x-7y+1=0$: (ii.) $x-y+36=0$, $7x+7y+12=0$.
5. $7x+9y-4=0$, $7x-17y-38=0$, $56x+7y-117=0$, $\left(\frac{205}{91}, \frac{-17}{13}\right)$.
6. (i.) $14x-21y+2=0$, $3x+4y-7=0$, $17x-17y+5=0$.
(ii.) $21x+14y-10=0$, $4x-3y-1=0$, $17x+17y-9=0$,
$\left(\frac{44}{119}, \frac{19}{119}\right)$. **7.** $7x-9y=0$, $2x=3$, $9x+33y=52$.
8. $x-6y+8=0$. **9.** $(1, -2)$ $\left(\frac{53}{33}, \frac{-12}{11}\right)$.
10. $6x+8y-25=0$, $6x+8y+15=0$.
11. $19x+3y+26=0$. **12.** $x+7y-2=0$, $7x-y=0$.

Examples XI. (p. 62)

1. $x+y=k$. **2.** $x+y=k$. **3.** $bx+ay=2k$.

4. (ii.) $\pm\frac{b}{a}$. **5.** $2x-4y=a-2b$.

6. A straight line, perpendicular from origin$=2$. (i.) A line at right angles through $(2, 3)$; (ii.) length of perpendicular from $(2, 3)$.

7. $(2\cos\theta, 3\cos\theta+2\sin\theta)$; $\frac{x^2}{4}+\left(\frac{y}{2}-\frac{3x}{4}\right)^2=1$.

8. $x-y=4$ if **Y** axis is positive upwards, origin **B**.

10. $\frac{\alpha}{x}+\frac{\beta}{y}=1$. 11. Locus $x+y=\frac{a}{2}$. 12. $\frac{1}{a}+\frac{1}{b}=\frac{1}{\mathrm{OA}}$.

13. $x+y=5$. 14. $\frac{\alpha}{2x}+\frac{\beta}{2y}=1$. 16. $ax-by+bp=0$.

17. $x^2-2xy-x(a+b)+y(a+b)+ab=0$. 18. $x^2-2x-4y+5=0$.

19. $\frac{4x-a}{2b}+\frac{4y-c}{2d}=1$ where $\mathrm{OD}=d$, $\mathrm{OC}=c$, etc.

20. $x=\frac{12\cos\theta+9\sin\theta}{5}$; $y=\frac{16\cos\theta+12\sin\theta}{5}$; $4x-3y=0$.

21. $x=0$. 22. $x^2-y^2=a^2$.

23. $2m^2x^2+y^2(3+m^2)-2m^2ax-2may+m^2a^2=\lambda(1+m^2)$.

24. $y-\frac{\beta}{2}=-\frac{\alpha}{\beta}\left(x-\frac{\alpha}{2}\right)$.

Miscellaneous Examples

A. (p. 65)

1. 9. 2. $3x+4y-5=0$; $x-y-4=0$ $(3, -1)$

3. $x\pm\frac{y}{\sqrt{2}}=1$; $2x\mp2y\sqrt{2}+1=0$; $90°$.

4. $16x^2+24xy+9y^2+12x-66y+21=0$.

5. $3x+4y-25=0$.

B. (p. 65)

1. $ax+by-a^2=0$. 2. $\left(\frac{ab}{a+b}\right)$, $\left(\frac{ab}{a+b}\right)$, $\frac{ab}{a+b}$.

3. $99x+27y-130=0$; or $3x-11y=0$.

4. $\left(\frac{4}{3}, 1\right)$. 5. $(-3, 0)$ $(2, 0)$ $(0, 6)$ $(0, -1)$; 5, 7.

C. (p. 66)

1. $2x+y=0$. 2. $\left(6\frac{1}{5}, 4\frac{5}{6}\right)$ $\left(3\frac{4}{5}, 3\frac{1}{6}\right)$.

3. $67° 22'$. 4. $\frac{\alpha+\beta+\gamma}{3}$. 5. $x+y-7=0$.

D. (p. 66)

1. $\frac{4}{5}x+\frac{3}{5}y=\frac{37}{5}$; $-\frac{4}{5}x-\frac{3}{5}y=\frac{13}{5}$; (7, 3) (4, 7); (−1, −3) (−4, 1).

2. $3x^2+3y^2+4x-16y+12=0$.

3. $\frac{12}{7}, \frac{23}{14}$.

4. $\frac{x}{2a}+\frac{y}{2b}=1$.

5. $3x-y-4=0$; $x+3y-8=0$; $(-4, -16)$.

E. (p. 67)

1. $\left(\frac{13}{5}, 0\right)$.

2. $y=2x\pm3\sqrt{5}$; $y=-\frac{1}{2}x\pm\frac{3}{\sqrt{5}}$.

3. $ay-bx=2k$; $90°$.

4. $\left(a, \frac{a^2-ax_1}{y_1}\right)$; $\left(-a, \frac{a^2+ax_1}{y_1}\right)$.

5. (x_1+2a), 0; $y^2=a(x-a)$.

F. (p. 67)

1. R $\frac{ab^2}{a^2+ab+b^2}, \frac{a^2b}{a^2+ab+b^2}$.

3. $\frac{8}{3}$.

4. $\left(\frac{2}{3}, \frac{4}{3}\right)$ $\left(\frac{1}{7}, \frac{13}{7}\right)$.

5. $(-4, 5)$ $\left(-\frac{194}{29}, -\frac{11}{29}\right)$.

G. (p. 68)

2. $y^2=(x-\mathrm{OA})(x-\mathrm{OB})$.

4. 3·73, 0·27; $y=3{\cdot}73(x-4)$; $y={\cdot}27x+4$.

5. $y=2x$, $x=0$.

H. (p. 68)

1. $\frac{48}{37}, \frac{36}{37}$.

3. $\left(1, \frac{4}{3}\right)$.

5. (3, 3).

CHAPTER V

Examples XII(*a*). (p. 72)

1. (i.) 6; (ii.) 1; (iii.) a^2+b^2; (iv.) −4;
(v.) $a_1(b_2-b_3)-b_1(a_2-a_3)+a_2b_3-a_3b_2$.

2. (i.) $\frac{8}{5}, -\frac{7}{10}$; (ii.) −1, 0; (iii.) $-\frac{13}{8}, \frac{3}{4}$; (iv.) $-\frac{1}{5}, \frac{13}{5}$.

Examples XII(*b*). (p. 74)

1. (i.) 6; (ii.) 2; (iii.) $\frac{1}{2}(a^2+b^2)$; (iv.) $\frac{3}{2}$; (v.) 6; (vi.) $\frac{19}{2}$.

3. (i.) $x-3y+7=0$; (ii.) $5x-4y-6=0$; (iii.) $x+6y-7=0$;
(iv.) $4x-y-6=0$; (v.) $bx-2ay+ab=0$; (vi.) $x+2y-4=0$.

CHAPTER VI

Examples XIII. (p. 80)

1. (i.) $x=0,\ y=0$; (ii.) $x=2,\ x=1$; (iii.) $x=4y,\ x=y$.
2. (i.) $x-4y+4=0,\ x-y-1=0,\ 31°$;
(ii.) $3x+2y-1=0,\ x+y-1=0,\ 11°\ 19'$;
(iii.) $x=b,\ y=a,\ 90°$.
3. (i.) $x^2+y^2-4x-2y+1=0$; (ii.) $x^2+y^2+6x-4y-3=0$.
4. (i.) $(-2, 0),\ 3$; (ii.) $(3, 0),\ \sqrt{13}$; (iii.) $(0, 0),\ 0$;
(iv.) $\left(\frac{1}{2}, -\frac{1}{2}\right),\ \frac{1}{\sqrt{2}}$; (v.) $\left(\frac{1}{2}, -\frac{1}{4}\right),\ 1$; (vi.) $\left(\frac{2}{7}, \frac{1}{14}\right),\ \frac{\sqrt{101}}{14}$.
5. (i.) $x^2+y^2-4x-6y=13$; (ii.) $x^2+y^2+5y-30=0$;
(iii.) $x^2+y^2+2x+10y+1=0$; (iv.) $5x^2-14x+5y^2+7y-24=0$.
6. (i.) $x^2-4x+y^2-6y+4=0$; (ii.) $x^2+4x+y^2-6y+4=0$.
7. (i.) $x^2+y^2=9$; (ii.) $x^2+y^2=32$.
8. (i.) $4x+3y=12$: (ii.) $\frac{4\alpha+3\beta-12}{5}$; (iii.) $(x-1)^2+(y-1)^2=1$.
9. $x^2+y^2=b^2-a^2$. A point circle at the origin.
10. $x^2+y^2=ky,\ \left(0, \frac{k}{2}\right)$. **11.** $x^2+y^2-8x-10y+1=0$.
12. $8x^2-15y^2-2xy+11y-2=0$; $8x^2-2xy-15y^2=0$.
14. $\left(\frac{1}{13}, \frac{5}{13}\right)\left(-\frac{2}{13}, \frac{3}{13}\right)\left(\frac{3}{13}, \frac{2}{13}\right)$ $(0, 0)$. **15.** $x^2-y^2=l^2$.
16. $x^2-x(x_1+x_2)+y^2-y(y_1+y_2)+x_1x_2+y_1y_2=0$.
17. $2x+y-4=0,\ x-2y+3=0,\ 2x^2-3xy-2y^2=0$.
18. $x+2y=0$. **20.** $(3, 3)\ (1, 1)$. **21.** 41.
23. $62°\ 43'$. **24.** $\sqrt{19}$.

Examples XIV. (p. 89)

1. (i.) $y^2=4(x-1)$; (ii.) $y^2=16x$; (iii.) $x^2=8(y-2)$;
(iv.) $(4x+3y)^2-20x-140y+100=0$; (v.) $y^2=-8x$;
(vi.) $x^2=-8y$; (vii.) $x^2=-8(y-2)$.
2. (i.) $(y-2)^2=8x$; (ii.) $(y-2)^2=-4(x+1)$;
(iii.) $(x-1)^2=-4(y-3)$; (iv.) $(x-1)^2=12(y+1)$.
3. (i.) $\mathsf{A}\left(-\frac{1}{2}, -\frac{1}{4}\right)\ \mathsf{S}\left(-\frac{1}{2}, 0\right),\ 2y+1=0$;
(ii.) $\mathsf{A}\left(\frac{1}{3}, -\frac{1}{3}\right)\ \mathsf{S}\left(\frac{1}{3}, -\frac{1}{4}\right),\ 12y+5=0$;
(iii) $\mathsf{A}(-1, 1)\ \mathsf{S}\left(-\frac{7}{8}, 1\right),\ 8x+9=0$;

(iv.) A $\left(-\frac{7}{8}, \frac{1}{2}\right)$ S $\left(-\frac{11}{8}, \frac{1}{2}\right)$, $8x+3=0$.

(v.) A $(-3, -5)$ S $\left(-3, -\frac{19}{4}\right)$, $4y+21=0$.

4. $4x-y-4=0$, $(0, -4)$. $SP=\frac{17}{4}$.

5. $x+y-3=0$, G(3, 0). NG$=2$.

7. $y^2+x=0$, $4x^2-y=0$.

8. 31·25; 5·4, 6·6, 8·6, 11·4.

9. 9·6, 8·4.

10. $a=o$, $b=1$, $c=-\frac{1}{20}$; 5 ft.; 3·75 ft.

11. 8° 32′; 6 ft.

13. $\left(-\frac{\kappa}{2\lambda}, -\frac{\kappa^2}{4\lambda}\right)$.

15. (i.) $(4x+3y)^2-38x-66y+46=0$. Axis $y-1=-\frac{4}{3}(x-1)$.

(ii.) $(3x-4y)^2-60x+80y+100=0$. Axis $3x-4y-10=0$.

16. $\left(\frac{71}{25}, \frac{22}{25}\right)$; $\frac{14}{5}$.

Examples XV. (p. 99)

1. (i.) $\frac{x^2}{16}+\frac{y^2}{4}=1$; (ii.) $\frac{x^2}{4}+\frac{y^2}{16}=1$.

2. $\frac{(x-2)^2}{16}+\frac{(y-1)^2}{4}=1$; $e=\frac{\sqrt{3}}{2}$.

3. (i.) $(2, -2)$; (ii.) $(-2, 2)$; (iii.) $(1, 2)$.

4. (i.) $(2, 0·7)$ $(2, -4·7)$; (ii.) $(-4·7, 2)$ $(0·7, 2)$; (iii.) $(4·78, 2)$ $(-2·78, 2)$.

5. $91x^2+84y^2-24xy-170x-360y+475=0$.

6. (i.) $\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}=1$; (ii.) $\frac{(x+2)^2}{9}+\frac{(y+3)^2}{4}=1$.

7. (i.) $\frac{x^2}{4}+\frac{y^2}{9}=1$; (ii.) $2x^2+4y^2=11$.

8. $\frac{x^2}{16}+\frac{y^2}{7}=1$.

9. $PC_1+PC_2=r_2-r_1=$const.

10. 0·66.

11. 10·47″, 4·47″.

12. (i.) $3x^2+5y^2=32$; (ii.) $\frac{x^2}{81}+\frac{y^2}{45}=1$; (iii.) $\frac{x^2}{144}+\frac{y^2}{128}=1$; (iv.) $\frac{x^2}{25}+\frac{y^2}{9}=1$.

13. $\frac{x^2}{9}+\frac{y^2}{4}=1$; 6π.

14. 72π.

15. $SP=a$; $ae=0$ $\therefore e=0$; $\frac{a}{e}=\infty$.

16. $1:\frac{b}{a}$.

17. $SB=a=3$.

18. $\frac{x^2}{2}+\frac{(y-1)^2}{4}=1$.

19. $x=a\cos\theta, y=b\sin\theta$.

20. $\sqrt{\frac{2}{3}}$.

Examples XVI. (p. 104)

1. $\pm 2{\cdot}04,\ 0.$ 2. (i.) $6,\ \left(1,-\frac{1}{2}\right),\ 1{\cdot}2$; (ii.) $5{\cdot}29,\ (-2,\ 1),\ 2{\cdot}24.$
3. $x^2+y^2+6xy-16x-8y+8=0.$
4. $r_1-r_2=\kappa.$ 6. (i.) $1{\cdot}34$; (ii.) $1{\cdot}25.$
7. (i.) $\frac{x^2}{9}-\frac{y^2}{4}=1$; (ii.) $\frac{x^2}{36}-\frac{4y^2}{25}=1$; (iii.) $\frac{x^2}{16}-\frac{y^2}{20}=1.$
8. $\sqrt{2}.$ 9. SP. $\mathrm{S}^1\mathrm{P}=e\left(x-\frac{a}{e}\right)e\left(x+\frac{a}{e}\right).$
11. (i.) Two lines; (ii.) circle; (iii.) rectangular hyperbola.
12. $\mathrm{AP}-\mathrm{DP}=2.$ 13. $y=\pm\frac{bx}{a}\left(1-\frac{a^2}{x^2}\right)^{\frac{1}{2}}.$
14. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$ 15. AN. $\mathrm{A}^1\mathrm{N}=(a+x_1)\ (x_1-a).$

Examples XVII. (p. 106)

1. $x^2+y^2=a^2.$
2. (i.) $3x-4y+5=0$; (ii.) $8a^2x=y^3$; (iii.) $xy=c^2.$
3. $x^2+y^2=a^2+b^2.$ 4. $y^2-2y-x+2=0.$
5. $x^2-6x+y^2-8y+21=0$; $(3,\ 4)$; $2.$
6. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$ 8. $x^2-y^2=4.$ 9. $x^3+y^3=3axy.$
10. $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$ 11. $x^2-y^2=1.$
12. $\mathrm{ON}=b\cot\theta,\ \mathrm{PN}=a\cos\theta\sin\theta.$

Examples XVIII. (p. 110)

1. Origin at centre of circle and $x=\kappa$ the line; the locus is a parabola $y^2=\lambda^2-2x\lambda$ where $\lambda=\kappa+r.$
2. $y^2=2\mathrm{AS}\left(x-\frac{\mathrm{AS}}{2}\right)$ (origin **A**). 3. $x^2+\frac{y^2}{\kappa}=\frac{a^2}{4}.$
4. $\frac{4x^2}{a^2}+\frac{(2y-b)^2}{b^2}=1.$ 5. $x=\frac{\kappa}{2e}$ (origin **X**).
6. $y^2(m+l)^2=4am^2x.$ 7. $y^2=2ax.$
8. $\frac{1}{(y-1)^2}=1+\frac{y^2}{x^2}.$ 9. $\left(\frac{x}{a}\right)^2+\left(\frac{y+b}{a}\right)^2=1.$
10. $\frac{x^2}{\mathrm{R}^2}+\frac{y^2}{r^2}=1$; $\frac{x^2}{r^2}+\frac{y^2}{\mathrm{R}^2}=1.$ 11. $x^2-y^2-50y=0$; $\sqrt{2}.$
12. $\frac{x^2}{(8-a)^2}+\frac{y^2}{a^2}=1$; $e=3{\cdot}7.$ 13. $y(m+n-2\kappa)=x(2mn-m\kappa-n\kappa).$
14. $2xy=\kappa(x^2-a^2).$ 15. $x^2+y^2-2xa\frac{(1+\lambda^2)}{1-\lambda^2}+a^2=0$;

$\lambda=1,\ x=0$; $\lambda=2.\ x^2+y^2+\frac{10}{3}ax+a^2=0.$

CHAPTER VII

Examples XIX. (p. 115)

1. $2x \pm 3y = 13$.
2. $5x - 6y = 0$.
3. $3x - 2y + 13 = 0$, $2x + 3y = 0$.
4. (i.) $3x - 4y + 8 = 0$, $12x + 9y - 68 = 0$;
   (ii.) $\frac{xx_1}{9} + \frac{yy_1}{4} = 1$, $9xy_1 - 4yx_1 - 5x_1y_1 = 0$;
   (iii.) $3x + 5y = 14$, $5x - 3y - 12 = 0$;
   (iv.) $3x + y - 4 = 0$, $x - 3y + 2 = 0$.
5. $3xx_1^2 - 8y - 16y_1 = 0$.
6. $(0, 0)$ $(\pm 2, \pm 4)$. At $\pm 2, \pm 4$, $\frac{dy}{dx} = 6$.
8. 200; 45°; 44 ft.
10. $\frac{2}{3}$.
11. $x - y + 2 = 0$.
12. $-\frac{b}{a}$.

Examples XX. (p. 119)

2. $\frac{8}{3}$.
3. (i.) $x - 2y + 4 = 0$; (ii.) $4x + 3y \pm 20 = 0$; (iii.) $y = -x \pm \sqrt{27}$.
4. $\kappa = \pm \frac{5r}{4}$.
5. $y = \pm \sqrt{3}x \pm \sqrt{13}$.
6. $\pm \sqrt{\frac{10}{3}}$.
8. $\pm \sqrt{\frac{a^2 - b^2}{2}}$.
11. $\pm \frac{a^2}{\sqrt{a^2 + b^2}}$, $\pm \frac{b^2}{\sqrt{a^2 + b^2}}$.
12. $\pm \frac{16}{5}$, $\pm \frac{9}{5}$, 50.
13. $y = -\frac{3x}{4} - \frac{7}{4}$; $y = -\frac{3x}{4} + \frac{13}{4}$.
14. $(3a, \pm 2a\sqrt{3})$; $\left(\frac{a}{3}, \pm \frac{2a}{\sqrt{3}}\right)$.
15. $y = \frac{1}{4}x + 9$; $y = \frac{9x}{4} + 1$.
16. $x = 1$ tangent; $x = 2$ real points.
17. $y = \pm x \pm \sqrt{a^2 + b^2}$.
19. at_1t_2, $a(t_1 + t_2)$.

Examples XXI. (p. 128)

1. $y = 4$.
2. $4x + 9y = 0$.
3. (i.) $\left(\frac{7}{25}, \frac{2}{5}\right)$; (ii.) $\left(\frac{502}{25}, \frac{2}{5}\right)$.
4. $9x - 5y = 0$.
5. $\left(\frac{3}{4}, \frac{9}{4}\right)$.
6. (i.) $\left(\frac{6}{13}, \frac{8}{13}\right)$; (ii.) $y = -3x \pm \frac{\sqrt{117}}{2}$.

8. $PQ=\frac{b}{a}x_1-y_1$. 10. $(x-\alpha)(y-\beta)=\kappa$.

11. (3, 2). 12. $x+2y=4$, $xy-2x+y=4$; $x+2y=7$.

13. $x^2+16xy=9$. 14. $x+yt^2=2ct$. 15. $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.

17. $\tan\alpha=\frac{b}{a}$. 19. $\frac{a^2+b^2}{4}$. 20. $y=\frac{b^2x}{a^2m}$.

23. Draw two sets of parallel chords, the loci of their mid points meet at the centre. Draw a circle and bisect common chords.

24. $m=\frac{2a}{y_1}\rightarrow 0$; $m=\frac{b^2}{a}\sqrt{\frac{1}{b^2}+\frac{1}{y_1^2}}\rightarrow\frac{b}{a}$.

CHAPTER VIII

Examples XXII. (p. 136)

4. $ax^3-x^2y^2+2axy^2-y^4=0$. 5. $y^2=16a(x+2a)$.

6. $y^2=4a(x-3a)$. 8. $y=\frac{x^2}{2}$.

15. Chord $y=mx+c$, $\mathbf{V}(\mathbf{X}, \mathbf{Y})$, $\mathbf{VG}$ is $y-\mathbf{Y}=-\frac{1}{m}(x-\mathbf{X})$.

$\therefore$ $\mathbf{AG}=m\mathbf{Y}+\mathbf{X}=2a+\mathbf{X}$. 17. 4.

Examples XXIII. (p. 140)

4. $\frac{4x^2}{25}+\frac{4y^2}{9}=1$; $\pm\frac{5}{2}$, 0; 0, $\pm\frac{3}{2}$.

6. $\frac{x^2}{36}+\frac{y^2}{32}=1$; $3\sqrt{2}$, 4. 9. 16.

12. Produce **SY** to meet **S'P'** in **R**. **QSY=YSP'=P'RY.**

14. $(x^2+y^2)^2=a^2x^2+b^2y^2$. 16. 4, 2·6.

Examples XXIV. (p. 144)

7. $\frac{(2x-ae)^2}{a^2}-\frac{4y^2}{b^2}=1$.

9. Join **PS**, draw **SZ** perpendicular to **SP**.

Miscellaneous Examples

I. (p. 145)

1. $x^2+4x+y^2-6y+8=0$; $2x+y-4=0$.

2. $x^2=4a(y-a)$; $x^2=4ay$. 3. See. p. 131.

4. $\frac{4x^2}{a^2}+\frac{(2y-b)^2}{b^2}=1$. 5. $y=mx+\frac{a}{m}$.

J. (p. 145)

1. $x^2-6x+y^2-8y+8\cdot4=0$.
2. $y=\sqrt{3}x+\frac{1}{\sqrt{3}}$; $\left(\frac{1}{3}, \frac{2}{\sqrt{3}}\right)$.
3. $y=-\frac{x}{m}\pm\frac{r\sqrt{1+m^2}}{m}$.
4. $\left(t, \frac{2}{t+1}\right)$; $y=\frac{2}{x+1}$.
5. 0·74.

K. (p. 146)

2. $\frac{5}{3}$, Q(4, 4), R(1, −1). $\Delta=4$.
3. $\frac{10}{19}$, 2.
5. 2″, 0·76.

L. (p. 146)

1. Draw circle **PSR** on **PR** as diameter. Draw **YS** perpendicular to **RP**. **SP**=4·9; **SZ**=3·4.
2. $x=x_1\cos 30-y_1\sin 30$; $y=y_1\cos 30+x_1\sin 30$; $\sqrt{3}(x_1^2-y_1^2)+2x_1y_1=4c$.
3. $-\frac{7}{4}$.
4. $9x^2y^2=(x^2+y^2)^3$.
5. 4·1.

M. (p. 147)

1. Draw common tangent to circles centre **P**, radius 2·5 and centre **Q** radius 5. 3·9.
2. (i.) $\left(\frac{5}{2}, 0\right)$; (ii.) (·56, 0) (4·44, 0); (iii.) 5; (iv.) ·78.
3. $27x=0$. Cusp at $x=2$. Asymptote $x=0$
4. $(2x-y+3)(x+2y-1)=0$. 90°.
5. $\frac{1}{4}(2^{q-p}-2^{p-q})$. $x^2-y^2=1$.

N. (p. 148)

1. $x^2-2x-4y+1=0$.
2. $xy=\frac{ah}{2}$.
3. $t_1t_2=-1$.
5. Divide **SH** externally at **K** in ratio 3 : 1 and draw circle on **GK** as diameter. **SP**=1·98.

O. (p. 148)

1. $\left(\frac{3t}{4}, \frac{t^2}{2}\right)$.
2. $5x-sy+10y-100=0$; $xs+20y-200=0$; $x^2-4y^2-20x+40y=0$.
4. $56x^2+119y^2+216xy-224x-332y+124=0$; $\left(\frac{23}{25}, \frac{14}{25}\right)$.
5. $x+4=0, y+3=0, xy=8$.

P. (p. 149)

1. $x-3y+1=0$, $3x+y-7=0$; $(11, 4)$ $(-1, 10)$.
2. $3y^2-2x^2-5xy=0$. Two straight lines.
4. Equation $y=5+\frac{x^2}{250}$; 6·6; 40 ft. 5. 12 ft.

Q. (p. 149)

2. $\tan a=\frac{16}{15}$.

CHAPTER IX

Examples XXV. (p. 160)

2. (i.) $\tan\theta=\frac{1}{2}$; (ii.) $r\sin\theta+4=0$; (iii.) $r=2$;
(iv.) $r^2\cos 2\theta=2$; (v.) $r^2\sin 2\theta=12$; (vi.) $r=2\cos\theta$.
3. (i.) $y=x\tan a$; (ii.) $(x^2+y^2)^2=a^2(x^2-y^2)$;
(iii.) $2x+y=l$; (iv.) $x^2+y^2=ax+by$.
6. Take origin at end of a diameter $r=2d\cos\theta$.
7. Take line from point **O** to centre **C** as initial line. If $\mathbf{OC}=c$ and radius $=a$, $\frac{a^2}{4}=r^2+c^2-2rc\cos\theta$.
8. (i.) $r=2\operatorname{cosec}\theta$; (ii.) $r=2\sec\theta$.
10. $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$; $\frac{dy}{dx}=\tan\theta$.
11. $x(y^2+a^2)=a^3$. 12. $r=2\cos 45\cos(\theta-45)$. Centre $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$ or in Cartesians $x^2-x+y^2-y=0$.

CHAPTER X

Examples XXVI. (p. 165)

1. $\sqrt{26}$, $\sqrt{21}$. 2. 3, 5, $\sqrt{13}$. 3. $\sqrt{18}$, $\sqrt{46}$, $\sqrt{3\cdot 5}$.
4. $\cos a=\frac{1}{\sqrt{3}}$, $a=54°\ 44'$. 5. 60°. 6. In plane **XOZ**.
7. 77° 24′, 115° 54′, 29° 12′. 8. $\frac{1}{3}$, $-\frac{2}{3}$, $\frac{2}{3}$.
9. $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, 0; $\frac{1}{\sqrt{2}}$, 0, $\frac{1}{\sqrt{2}}$.
10. 0, $\frac{4}{\sqrt{52}}$, $\frac{6}{\sqrt{52}}$; $\frac{5}{\sqrt{77}}$, $\frac{4}{\sqrt{77}}$, $\frac{6}{\sqrt{77}}$.

Examples XXVII. (p. 169)

1. $\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}; \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$; 70° 54′.

2. 3, 2, 4, $\sqrt{13}$, $\sqrt{20}$, 5.

3. 4, 1, 2, $\sqrt{17}$, $\sqrt{5}$, $\sqrt{26}$.

4. 60°.

5. $\frac{5}{\sqrt{77}}, \frac{6}{\sqrt{77}}, \frac{4}{\sqrt{77}}$, 34° 45′.

6. $\left(4, \frac{5}{2}, \frac{11}{2}\right)$.

7. $\left(\frac{4}{3}, \frac{4}{3}, \frac{1}{3}\right)$.

8. 4, 5, 0.

9. $\frac{lx_2-mx_1}{l-m}, \frac{ly_2-my_1}{l-m}, \frac{lz_2-mz_1}{l-m}$.

10. $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$. 65° 54′, 57° 57′.

Examples XXVIII. (p. 176)

1. $2x-3y-z+6=0$; $-\frac{2x}{\sqrt{14}}+\frac{3y}{\sqrt{14}}+\frac{z}{\sqrt{14}}=\frac{6}{\sqrt{14}}$.

2. $x+y+z=3$.

3. $5x-7y+11z+4=0$.

4. 64° 37′, 106° 36′, 31°, 59°.

5. $7x-6y+z=0$; $x-2y=0$.

6. $y-3z=0$.

7. 5, $-\frac{10}{3}$, 2.

8. (2, 1, 3).

9. $\frac{20}{\sqrt{129}}$.

10. 53° 18′.

11. $2x-3z=6$;

12. 67° 25′.

13. $x-2y+3z=5$.

14. $x+3y+z=2\sqrt{11}$,

15. $4x-5y-z+16=0$; $5x-7y-13z+20=0$; 55° 18′. $\frac{5}{9\sqrt{3}}; \frac{-7}{9\sqrt{3}}; \frac{-13}{9\sqrt{3}}$.

16. $\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$.

17. 48° 59′; 8·2 ft.; 137·6.

18. $4\sqrt{3}$.

19. $\frac{1}{2}$ ft.; 36° 52′.

20. $2x-3z=0$; $y-z=0$; $4x+y+5z=24$; $x+2z=6$; $y+z=4$; $y+z=4$.

21. 12·87 ft. A (12, 0, 10), B(0, 9, 10), O(0, 0, 8), C(17·2, 9, 12·87); $3x+4y-18z=144$.

22. $3x+2y+6z=6$; $\frac{28}{3}$; $\frac{8}{3}$.

Examples XXIX. (p. 182)

1. $\frac{x-2}{1}=\frac{y-3}{-1}=\frac{z-1}{2}$.

2. $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z+1}{1}$.

3. $\frac{x-2}{1}=\frac{y-1}{3}=\frac{z+1}{1}$.

4. $\frac{4}{\sqrt{42}}, -\frac{1}{\sqrt{42}}, \frac{5}{\sqrt{42}}$

5. $\frac{x-3}{-5}=\frac{y-1}{2}=\frac{z-2}{2}$; $\frac{x+2}{5}=\frac{y}{2}=\frac{z-1}{-5}$; $(8, -1, 0)$ $(0, \frac{4}{5}, -1)$.

6. $31° 49'$; $58° 11'$. 7. 4. 8. $\frac{x-3}{3}=\frac{y-4}{0}=\frac{z-5}{-3}$.

9. (i.) $\left(-\frac{5}{2}, \frac{3}{2}, 0\right)$; (ii.) $(-7, 0, -3)$; (iii.) $\left(0, \frac{7}{3}, \frac{5}{3}\right)$;

(iv.) $(29, 12, 21)$; (v.) $\left(15\frac{1}{2}, 7\frac{1}{2}, 12\right)$.

10. $\frac{x-1}{-5}=\frac{y-1}{7}=\frac{z-5}{13}$. $(1{\cdot}97, -{\cdot}35, 2{\cdot}5)$.

11. $\frac{x+2}{2}-2=\frac{y+8}{5}-2=\frac{z+5}{3}-2$.

12. $\frac{x}{2}=-\frac{y}{9}=\frac{z}{13}$; $\frac{x+1}{-5}=\frac{y-4}{7}=\frac{z}{1}$.

13. $86\frac{2}{3}$; $15° 54'$. 14. $133° 49'$.

15. 198·2 W., 152·3 N., −325·6, 410·5 yds.; N. 52° 28′ W.; 37° 31′.

16. $(-5, -{\cdot}5, 0)$. 17. $(3\frac{1}{2}, 7\frac{1}{4}, 0)$ $(0, 19\frac{1}{2}, -10\frac{1}{2})$.

18. $3y+4z=12$; $\frac{3}{\sqrt{34}}, \frac{4}{\sqrt{34}}, \frac{-3}{\sqrt{34}}$; $\frac{x}{3}=\frac{y}{4}=\frac{z-3}{-3}$; $\frac{3}{\sqrt{34}}, \frac{-4}{\sqrt{34}}, \frac{3}{\sqrt{34}}$;

$\frac{x}{3}=\frac{y-4}{-4}=\frac{z}{3}$.

19. At $(1, -1, 2)$. $\frac{x-1}{1}=\frac{y+1}{2}=\frac{z-2}{2}$; $\frac{x-1}{-2}=\frac{y+1}{-1}=\frac{z-2}{3}$.

20. $\frac{x+\frac{33}{13}}{3}=\frac{4\left(y+\frac{11}{13}\right)}{3}=\frac{z}{1}$.

Examples XXX. (p. 187)

1. $\frac{71}{\sqrt{769}}=2{\cdot}56$. 2. $4x-3y+z=0$; $\frac{7}{\sqrt{26}}$.

3. $x-3y+2z=1$. 5. $(1, 0, 0)$. 6. $5y-2z=0$.

7. $\frac{120}{\sqrt{469}}=5{\cdot}54$; $\frac{9}{5}, \frac{10}{3}$. 8. $\frac{4}{\sqrt{14}}=1{\cdot}07$.

9. $3x-4y+7z+13=0$. 10. $25x-23y+2z=0$.

11. $4x+3y=24$. 12. $\frac{x-6}{3}=\frac{y}{-4}=\frac{z-3}{3}$.

13. $3x-y+2z=12\pm2\sqrt{14}$. 14. $3x-2y-6z+5=0$.

15. $\frac{13}{\sqrt{90}}=1{\cdot}37$.

Examples XXXI. (p. 195)

1. $z=6$, $y=4$. 2. $z=2$; $x^2+y^2+z^2-4x-2y-8z=-12$.
3. $3x+y+2z=0$; $x^2+y^2+z^2=16$.
4. $x^2-4x+y^2+z^2+2z=13$.
6. $y^2+z^2=(mx+b)^2$; $(y-b)^2=m^2(x^2+z^2)$.
9. Sphere, cylinder, cylinder.

Miscellaneous Examples

R. (p. 196)

1. 12. 2. $x=4$, $3y-5z=0$; $y=5$, $3x-4z=0$; $72°\ 1'$.
3. $6x+8y+12z=61$. 4. $6x+2y-4z=8$. 5. $\sqrt{61}$.

S. (p. 196)

1. $\frac{19}{7}$. 2. $\frac{x-1}{-3}=\frac{y+2}{5}=\frac{z-3}{-2}$; $120°\ 58'$.
3. $\frac{x}{p}+\frac{y}{q}=1$.
4. $(70, -70, 99)$ $(70, -70, -17)$, $(70, 263, 99)$, $(70, 263, -17)$; $x=70$, $z=99$; 38,000.
5. $2x+3y+6z=49$.

T. (p. 197)

1. $\left(5, 5, \frac{13}{2}\right)$; $(5, 6, 7)$. 2. $\frac{16}{\sqrt{677}}, \frac{14}{\sqrt{677}}, \frac{15}{\sqrt{677}}$. 3. $45°$, 5.
4. $x-5z+20=0$; above; $\frac{3}{2\sqrt{26}}=\cdot 29$. 5. 3.

U. (p. 197)

1. $2x-4y+5z=0$. 2. $2x+y=17$; 4; $\left(\frac{29}{5}, \frac{27}{5}\right)$; $36°\ 42'$.
3. $\frac{9}{\sqrt{490}}, \frac{3}{\sqrt{490}}, \frac{-20}{\sqrt{490}}$. 4. $\frac{1}{\sqrt{7}}, \frac{2\sqrt{3}}{\sqrt{28}}, \frac{2\sqrt{3}}{\sqrt{28}}$, $\sqrt{28}=5\cdot 29$.
5. 10; $y+2z=14$; $x-2z+14=0$.